XVII. *On a Class of Identical Relations in the Theory of Elliptic Functions.*

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§ 1. THE object of the present paper is to notice certain forms into which the series for the primary elliptic functions admit of being thrown, and to discuss the identical relations to which they give rise. These latter, it will be shown, may be obtained directly by the aid of FOURIER'S theorem, or in a less straightforward manner by ordinary algebra.

§ 2. Whenever we have a periodic function of x, say ψx, such that $\psi x=\psi(x+\mu)$, it is well known that we may assume, for all values of x,

$$\psi x=A_0+A_1\cos\frac{2\pi x}{\mu}+A_2\cos\frac{4\pi x}{\mu}+\&c.$$

$$+B_1\sin\frac{2\pi x}{\mu}+B_2\sin\frac{4\pi x}{\mu}+\&c.;$$

and if ψx be even, so that $\psi x=\psi(-x)$, then B_1, B_2, &c. all vanish; while if ψx is uneven, so that $\psi x=-\psi(-x)$, A_0, A_1, &c. vanish. If ψx is such that $\psi x=-\psi(x+\mu)$, then we have

$$\psi x=A_1\cos\frac{\pi x}{\mu}+A_3\cos\frac{3\pi x}{\mu}+\&c.$$

or

$$=B_1\sin\frac{\pi x}{\mu}+B_3\sin\frac{3\pi x}{\mu}+\&c.,$$

according as ψx is even or uneven.

But there is another totally different form in which ψx may generally be exhibited, viz.

$$\psi x=\phi x+\phi(x-\mu)+\phi(x+\mu)+\phi(x-2\mu)+\phi(x+2\mu)+\&c.$$

or

$$=\phi x-\phi(x-\mu)-\phi(x+\mu)+\phi(x-2\mu)+\phi(x+2\mu)-\&c.,$$

according as $\psi(x+\mu)=\psi x$ or $=-\psi x$.

The sine and cosine cannot be so expressed, but the other primary circular functions do admit of this form, as, *ex. gr.*, in the formulæ

$$\cot x=\frac{1}{x}+\frac{1}{x-\pi}+\frac{1}{x+\pi}+\frac{1}{x-2\pi}+\frac{1}{x+2\pi}+\&c.,$$

$$\operatorname{cosec} x=\frac{1}{x}-\frac{1}{x-\pi}-\frac{1}{x+\pi}+\frac{1}{x-2\pi}+\frac{1}{x+2\pi}-\&c.$$

(in which, after the first term, the series proceed by pairs of terms, so that for every term $\frac{1}{x-n\pi}$ there is a term $\frac{1}{x+n\pi}$).

Thus in general (although the sine and cosine are, as just mentioned, exceptions) we shall have, by equating the different forms of ψx, identities such as *ex. gr.* (if ψ is even)

$$\phi x+\phi(x-\mu)+\phi(x+\mu)+\text{ \&c.}=A_0+A_1\cos\frac{2\pi x}{\mu}+A_2\cos\frac{4\pi x}{\mu}+\text{ \&c.}$$

Also, it will be seen in § 10 that in certain cases even when ψx is not periodic it may be exhibited in the form $\phi x+\phi(x-\mu)+\phi(x+\mu)+$ &c., and we shall obtain identities in which the two sides of the equation are non-periodic.

§ 3. Before applying these principles to the elliptic functions, it is convenient to write down at once the following eight formulæ, which are to be found in the 'Fundamenta Nova' (pp. 101, 102, &c.), and which are all placed together in DURÈGE's 'Theorie der elliptischen Functionen' (Leipzig, 1861), pp. 226, 227:—

$$\sin\text{am}\,u=\frac{2\pi}{k\text{K}}\left\{\frac{q^{\frac{1}{2}}}{1-q}\sin\frac{\pi u}{2\text{K}}+\frac{q^{\frac{3}{2}}}{1-q^3}\sin\frac{3\pi u}{2\text{K}}+\text{ \&c.}\right\}, \quad . \quad . \quad . \quad (1)$$

$$\cos\text{am}\,u=\frac{2\pi}{k\text{K}}\left\{\frac{q}{1+q}\cos\frac{\pi u}{2\text{K}}+\frac{q^{\frac{3}{2}}}{1+q^3}\cos\frac{3\pi u}{2\text{K}}+\text{ \&c.}\right\}, \quad . \quad . \quad . \quad (2)$$

$$\Delta\text{ am}\,u=\frac{\pi}{2\text{K}}\left\{1+\frac{4q}{1+q^2}\cos\frac{\pi u}{\text{K}}+\frac{4q^2}{1+q^4}\cos\frac{2\pi u}{\text{K}}+\text{ \&c.}\right\}, \quad . \quad . \quad . \quad (3)$$

$$\tan\text{am}\,u=\frac{\pi}{2k'\text{K}}\left\{\tan\frac{\pi u}{2\text{K}}-\frac{4q^2}{1+q^2}\sin\frac{\pi u}{\text{K}}+\frac{4q^4}{1+q^4}\sin\frac{2\pi u}{\text{K}}-\text{ \&c.}\right\}, \quad . \quad . \quad . \quad (4)$$

$$\text{cosec am}\,u=\frac{\pi}{2\text{K}}\left\{\text{cosec}\frac{\pi u}{2\text{K}}+\frac{4q}{1-q}\sin\frac{\pi u}{2\text{K}}+\frac{4q^3}{1-q^3}\sin\frac{3\pi u}{2\text{K}}+\text{ \&c.}\right\}, \quad . \quad . \quad . \quad (5)$$

$$\sec\text{am}\,u=\frac{\pi}{2k'\text{K}}\left\{\sec\frac{\pi u}{2\text{K}}-\frac{4q}{1+q}\cos\frac{\pi u}{2\text{K}}+\frac{4q^3}{1+q^3}\cos\frac{3\pi u}{2\text{K}}-\text{ \&c.}\right\}, \quad . \quad . \quad . \quad (6)$$

$$\frac{1}{\Delta\text{ am}\,u}=\frac{\pi}{2k'\text{K}}\left\{1-\frac{4q}{1+q^2}\cos\frac{\pi u}{\text{K}}+\frac{4q^2}{1+q^4}\cos\frac{2\pi u}{\text{K}}-\text{ \&c.}\right\}, \quad . \quad . \quad . \quad (7)$$

$$\cot\text{am}\,u=\frac{\pi}{2\text{K}}\left\{\cot\frac{\pi u}{2\text{K}}-\frac{4q^2}{1+q^2}\sin\frac{\pi u}{\text{K}}-\frac{4q^4}{1+q^4}\sin\frac{2\pi u}{\text{K}}-\text{ \&c.}\right\}, \quad . \quad . \quad . \quad (8)$$

wherein, of course, $q=e^{-\frac{\pi\text{K}'}{\text{K}}}$.

In what follows, let $r=e^{-\frac{\pi\text{K}}{\text{K}'}}$, and take

$$\mu=\frac{\pi\text{K}'}{\text{K}},\quad \nu=\frac{\pi\text{K}}{\text{K}'},$$

so that

$$q=e^{-\mu},\quad r=e^{-\nu},\quad\text{and }\mu\nu=\pi^2.$$

Also let $x=\frac{\pi u}{2K}$ and $z=\frac{\pi u}{2K'}$, so that $z=\frac{\pi x}{\mu}=\frac{\nu x}{\pi}$.

§ 4. The process of transformation into the form

$$\phi x \pm \phi(x-\mu) \pm \phi(x+\mu) + \text{\&c.}$$

may be conveniently exhibited on (2); we have

$\cos \operatorname{am} \frac{2Kx}{\pi} = \cos \operatorname{am} u = \sec \operatorname{am}(ui, k')$, which, from (6),

$$=\frac{\pi}{2kK'}\left\{\frac{2}{e^z+e^{-z}}-\frac{4r}{1+r}\,\frac{e^z+e^{-z}}{2}+\frac{4r^3}{1+r^3}\,\frac{e^{3z}+e^{-3z}}{2}-\text{\&c.}\right\}$$

$$=\frac{\pi}{kK'}\left\{\frac{1}{e^z+e^{-z}}-(e^z+e^{-z})(r-r^2+r^3-\text{\&c.})+(e^{3z}+e^{-3z})(r^3-r^6+r^9-\text{\&c.})-\text{\&c.}\right\}$$

$$=\frac{\pi}{kK'}\left\{\frac{1}{e^z+e^{-z}}-\frac{re^z}{1+r^2e^{2z}}-\frac{re^{-z}}{1+r^2e^{-2z}}+\frac{r^2e^z}{1+r^4e^{2z}}+\frac{r^2e^{-z}}{1+r^4e^{-2z}}-\text{\&c.}\right\}$$

$$=\frac{\pi}{kK'}\left\{\frac{1}{e^z+e^{-z}}-\frac{1}{re^z+r^{-1}e^{-z}}-\frac{1}{r^{-1}e^z+re^{-z}}+\frac{1}{r^2e^z+r^{-2}e^{-z}}+\frac{1}{r^{-2}e^{2z}+r^2e^{-2z}}-\text{\&c.}\right\}$$

$$=\frac{\pi}{kK'}\left\{\frac{1}{r^{\frac{x}{\pi}}+r^{-\frac{x}{\pi}}}-\frac{1}{r^{\frac{x}{\pi}-1}+r^{-\left(\frac{x}{\pi}-1\right)}}-\frac{1}{r^{\frac{x}{\pi}+1}+r^{-\left(\frac{x}{\pi}+1\right)}}+\frac{1}{r^{\frac{x}{\pi}-2}+r^{-\left(\frac{x}{\pi}-2\right)}}+\frac{1}{r^{\frac{x}{\pi}+2}+r^{-\left(\frac{x}{\pi}+2\right)}}-\text{\&c.}\right\}. \quad (9)$$

The process requires that re^z should be <1, that is, that u should be $<2K$; but as both sides of the equation are such that they change sign without being altered in value when $u+2K$ is written for u, we see that the result obtained is true for all values of u. Thus we have

$$\cos \operatorname{am} 2Kx=\frac{\pi}{kK'}\left\{\frac{1}{r^x+r^{-x}}-\frac{1}{r^{x-1}+r^{-(x-1)}}-\frac{1}{r^{x+1}+r^{-(x+1)}}+\frac{1}{r^{x-2}+r^{-(x-2)}}+\text{\&c.}\right\} \quad . \quad . \quad (10)$$

for all values of x.

If in (10) we take $x=0$, we have

$$\frac{kK'}{\pi}=\tfrac{1}{2}-\frac{2}{r^{-1}+r}+\frac{2}{r^{-2}+r^2}-\text{\&c.};$$

or, writing K and k' for K′ and k, and therefore q for r,

$$\frac{2k'K}{\pi}=1-\frac{4q}{1+q^2}+\frac{4q^2}{1+q^4}-\text{\&c.},$$

which is at once seen to follow from (7), and is given by JACOBI, 'Fundamenta Nova,' p. 103.

It is, of course, easy to deduce (9) directly from the infinite product

$$\sqrt{\left(\frac{1-\cos \operatorname{am} u}{1+\cos \operatorname{am} u}\right)}=\tan\frac{x}{2}\prod_1^\infty\frac{(1-2q^{2n}\cos x+q^{4n})(1+2q^{2n-1}\cos x+q^{4n-2})}{(1+2q^{2n}\cos x+q^{4n})(1-2q^{2n-1}\cos x+q^{4n-2})};$$

for consider

$$\frac{1-2q^{2n}\cos x+q^{4n}}{1+2q^{2n}\cos x+q^{4n}}, \text{ which } =\frac{(1-q^{2n}e^{ix})(1-q^{2n}e^{-ix})}{(1+q^{2n}e^{ix})(1+q^{2n}e^{-ix})}.$$

Taking the logarithm and differentiating, we obtain, after a little reduction,

$$\frac{\pi i}{\mathrm{K}}\left\{\frac{1}{q^{2n}e^{ix}-q^{-2n}e^{-ix}}+\frac{1}{q^{-2n}e^{ix}-q^{2n}e^{-ix}}\right\}.$$

Similarly, from the uneven factor we get

$$-\frac{\pi i}{\mathrm{K}}\left\{\frac{1}{q^{2n-1}e^{ix}-q^{-(2n-1)}e^{-ix}}+\frac{1}{q^{-(2n-1)}e^{ix}-q^{2n-1}e^{-ix}}\right\};$$

thus

$$\frac{k'}{\cos\operatorname{am}(\mathrm{K}-u)}=\frac{\pi}{2\mathrm{K}}\operatorname{cosec} x+\frac{\pi i}{\mathrm{K}}\Sigma\left\{\frac{1}{q^{2n}e^{ix}-q^{-2n}e^{-ix}}+\frac{1}{q^{-2n}e^{ix}-q^{2n}e^{-ix}}-\frac{1}{q^{2n-1}e^{ix}-q^{-(2n-1)}e^{-ix}}-\frac{1}{q^{-(2n-1)}e^{ix}-q^{2n-1}e^{-ix}}\right\}.$$

Replace u by $\mathrm{K}-u$, that is to say x by $\frac{1}{2}\pi-x$, and remembering that $e^{\frac{1}{2}i\pi}=i$, $e^{-\frac{1}{2}i\pi}=-i$, we find

$$\sec\operatorname{am} u=\frac{\pi}{2k'\mathrm{K}}\left\{\sec x+2\Sigma\left(\frac{1}{q^{2n}e^{ix}+q^{-2n}e^{-ix}}+\ldots-\frac{1}{q^{2n-1}e^{ix}+q^{-(2n-1)}e^{-ix}}-\ldots\right)\right\}.$$

Herein write ui for u and k' for k, and we obtain the value on the right-hand side of (9) for sec am (ui, k'), that is, for cos am u.

If the other formulæ in the group (1) to (8) be transformed in the same way, viz. by use of the identical equations

$$\sin\operatorname{am} u=-i\tan(ui, k'),$$

$$\Delta\operatorname{am} u=\operatorname{cosec}\operatorname{am}(ui+\mathrm{K}', k'),$$

we obtain the following seven formulæ:—

$$\sin\operatorname{am} 2\mathrm{K}x=-\frac{\pi}{2k\mathrm{K}'}\left\{\frac{r^x-r^{-x}}{r^x+r^{-x}}-\frac{r^{x-1}-r^{-(x-1)}}{r^{x-1}+r^{-(x-1)}}-\frac{r^{x+1}-r^{-(x+1)}}{r^{x+1}+r^{-(x+1)}}+\frac{r^{x-2}-r^{-(x-2)}}{r^{x-2}+r^{-(x-2)}}+\frac{r^{x+2}-r^{-(x+2)}}{r^{x+2}+r^{-(x+2)}}-\&c.\right\}, \quad (11)$$

$$\Delta\operatorname{am} 2\mathrm{K}x=\frac{\pi}{\mathrm{K}'}\left\{\frac{1}{r^x+r^{-x}}+\frac{1}{r^{x-1}+r^{-(x-1)}}+\frac{1}{r^{x+1}+r^{-(x+1)}}+\&c.\right\}, \quad (12)$$

$$\tan\operatorname{am} 2\mathrm{K}x=\frac{\pi}{k'\mathrm{K}'}\left\{\frac{1}{r^{x-\frac{1}{2}}-r^{-(x-\frac{1}{2})}}+\frac{1}{r^{x+\frac{1}{2}}-r^{-(x+\frac{1}{2})}}+\frac{1}{r^{x-\frac{3}{2}}-r^{-(x-\frac{3}{2})}}+\&c.\right\}, \quad (13)$$

$$\frac{1}{\sin\operatorname{am} 2\mathrm{K}x}=-\frac{\pi}{2\mathrm{K}'}\left\{\frac{r^x+r^{-x}}{r^x-r^{-x}}-\frac{r^{x-1}+r^{-(x-1)}}{r^{x-1}-r^{-(x-1)}}-\frac{r^{x+1}+r^{-(x+1)}}{r^{x+1}-r^{-(x+1)}}+\&c.\right\}, \quad (14)$$

$$\frac{1}{\cos\operatorname{am} 2\mathrm{K}x}=\frac{\pi}{k'\mathrm{K}'}\left\{\frac{1}{r^{x-\frac{1}{2}}-r^{-(x-\frac{1}{2})}}-\frac{1}{r^{x+\frac{1}{2}}-r^{-(x+\frac{1}{2})}}-\frac{1}{r^{x-\frac{3}{2}}-r^{-(x-\frac{3}{2})}}+\&c.\right\}, \quad (15)$$

$$\frac{1}{\Delta\operatorname{am} 2\mathrm{K}x}=\frac{\pi}{k'\mathrm{K}'}\left\{\frac{1}{r^{x-\frac{1}{2}}+r^{-(x-\frac{1}{2})}}+\frac{1}{r^{x+\frac{1}{2}}+r^{-(x+\frac{1}{2})}}+\frac{1}{r^{x-\frac{3}{2}}+r^{-(x-\frac{3}{2})}}+\&c.\right\}, \quad (16)$$

$$\cot\operatorname{am} 2\mathrm{K}x=-\frac{\pi}{\mathrm{K}'}\left\{\frac{1}{r^x-r^{-x}}+\frac{1}{r^{x-1}-r^{-(x-1)}}+\frac{1}{r^{x+1}-r^{-(x+1)}}+\&c.\right\}. \quad (17)$$

It must be remarked that in (11) and (14) the number of terms must always be uneven; this point will be noticed at greater length further on (§ 10).

§ 5. Writing the hyperbolic sine, cosine, &c. as sinh, cosh, &c., these formulæ may also be written in a somewhat different form: thus

$$\cos \operatorname{am} u = \frac{\pi}{2k\mathrm{K}'}\left\{ \operatorname{sech} \frac{\pi u}{2\mathrm{K}'} - \operatorname{sech} \frac{\pi}{2\mathrm{K}'}(u-2\mathrm{K}) - \operatorname{sech} \frac{\pi}{2\mathrm{K}'}(u+2\mathrm{K}) + \&\text{c.}\right\},$$

$$\sin \operatorname{am} u = \frac{\pi}{2k\mathrm{K}'}\left\{ \tanh \frac{\pi u}{2\mathrm{K}'} - \tanh \frac{\pi}{2\mathrm{K}'}(u-2\mathrm{K}) - \tanh \frac{\pi}{2\mathrm{K}'}(u+2\mathrm{K}) + \&\text{c.}\right\},$$

and similarly for the others.

I do not think it likely that the formulæ (10) to (17) are new, but I have not succeeded in finding them anywhere. SCHELLBACH ('Die Lehre von den elliptischen Integralen . . .' Berlin, 1864, p. 33) gives the corresponding forms for θu, $\theta_1 u$, &c., but he does not allude to the similar expressions for the elliptic functions. It would, however, in any case have been necessary for the explanation of the rest of this paper to have written down the latter and demonstrated one of them.

§ 6. By equating the values of sin am u, cos am u, &c., as given by (1) to (8) and by (10) to (17), we obtain a series of identities of an algebraical character (*i. e.* which are independent of the notation of elliptic functions). Thus from (2) and (10) we have (remembering the definitions of μ, ν, &c. at the end of § 3)

$$\frac{\pi}{k\mathrm{K}}\left\{\frac{\cos x}{\cosh\frac{\mu}{2}} + \frac{\cos 3x}{\cosh\frac{3\mu}{2}} + \frac{\cos 5x}{\cosh\frac{5\mu}{2}} + \&\text{c.}\right\} = \frac{\pi}{2k\mathrm{K}'}\{\operatorname{sech} z - \operatorname{sech}(z-\nu) - \operatorname{sech}(z+\nu) + \&\text{c.}\},$$

viz.

$$\frac{\cos x}{\cosh\frac{\mu}{2}} + \frac{\cos 3x}{\cosh\frac{3\mu}{2}} + \frac{\cos 5x}{\cosh\frac{5\mu}{2}} + \&\text{c.} = \frac{\pi}{2\mu}\left\{\operatorname{sech}\frac{\pi x}{\mu} - \operatorname{sech}\frac{\pi}{\mu}(x-\pi) - \operatorname{sech}\frac{\pi}{\mu}(x+\pi) + \&\text{c.}\right\}.$$

This may be written (by interchanging x and z, μ and ν) in the rather more convenient form

$$\operatorname{sech} x - \operatorname{sech}(x-\mu) - \operatorname{sech}(x+\mu) + \operatorname{sech}(x-2\mu) + \operatorname{sech}(x+2\mu) - \&\text{c.}$$

$$= \frac{2\pi}{\mu}\left\{\frac{\cos\frac{\pi x}{\mu}}{\cosh\frac{\pi^2}{2\mu}} + \frac{\cos\frac{3\pi x}{\mu}}{\cosh\frac{3\pi^2}{2\mu}} + \frac{\cos\frac{5\pi x}{\mu}}{\cosh\frac{5\pi^2}{\mu}} + \&\text{c.}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In the same way, by comparing (1) and (11), we find

$$\tanh x - \tanh(x-\mu) - \tanh(x+\mu) + \&\text{c.} = \frac{2\pi}{\mu}\left\{\frac{\sin\frac{\pi x}{\mu}}{\sinh\frac{\pi^2}{2\mu}} + \frac{\sin\frac{3\pi x}{\mu}}{\sinh\frac{3\pi^2}{2\mu}} + \&\text{c.}\right\}; \quad . \quad . \quad . \quad (19)$$

and by comparing (3) and (12),

$$\operatorname{sech} x + \operatorname{sech}(x-\mu) + \operatorname{sech}(x+\mu) + \&\text{c.} = \frac{\pi}{\mu}\left\{1 + \frac{2\cos\frac{2\pi x}{\mu}}{\cosh\frac{\pi^2}{\mu}} + \frac{2\cos\frac{4\pi x}{\mu}}{\cosh\frac{2\pi^2}{\mu}} + \&\text{c.}\right\}. \quad . \quad . \quad (20)$$

The comparison of (4) and (13) gives

$$\operatorname{cosech}\left(x-\frac{\mu}{2}\right)+\operatorname{cosech}\left(x+\frac{\mu}{2}\right)+\operatorname{cosech}\left(x-\frac{3\mu}{2}\right)+\operatorname{cosech}\left(x+\frac{3\mu}{2}\right)+\&c.$$

$$=\frac{\pi}{\mu}\left\{-\tan\frac{\pi x}{\mu}+\frac{4\sin\frac{2\pi x}{\mu}}{e^{\frac{2\pi^2}{\mu}}+1}-\frac{4\sin\frac{4\pi x}{\mu}}{e^{\frac{4\pi^2}{\mu}}+1}+\&c.\right\},$$

which, on replacing x by $x+\frac{1}{2}\mu$, becomes

$$\operatorname{cosech} x+\operatorname{cosech}(x-\mu)+\operatorname{cosech}(x+\mu)+\&c.=\frac{\pi}{\mu}\left\{\cot\frac{\pi x}{\mu}-\frac{4\sin\frac{2\pi x}{\mu}}{e^{\frac{2\pi^2}{\mu}}+1}-\frac{4\sin\frac{4\pi x}{\mu}}{e^{\frac{4\pi^2}{\mu}}+1}-\&c.\right\}. \quad (21)$$

From (5) and (14) we deduce

$$\coth x-\coth(x-\mu)-\coth(x+\mu)+\&c.=\frac{\pi}{\mu}\left\{\operatorname{cosec}\frac{\pi x}{\mu}+\frac{4\sin\frac{\pi x}{\mu}}{e^{\frac{\pi^2}{\mu}}-1}+\frac{4\sin\frac{3\pi x}{\mu}}{e^{\frac{3\pi^2}{\mu}}-1}+\&c.\right\}. \quad (22)$$

The comparison of (6) and (15) gives

$$-\operatorname{cosech}\left(x-\frac{\mu}{2}\right)+\operatorname{cosech}\left(x+\frac{\mu}{2}\right)+\operatorname{cosech}\left(x-\frac{3\mu}{2}\right)-\&c.$$

$$=\frac{\pi}{\mu}\left\{\sec\frac{\pi x}{\mu}-\frac{4\cos\frac{\pi x}{\mu}}{e^{\frac{\pi^2}{\mu}}+1}+\frac{4\cos\frac{3\pi x}{\mu}}{e^{\frac{3\pi^2}{\mu}}+1}-\&c.\right\},$$

which, on replacing x by $x+\frac{1}{2}\mu$, becomes

$$\operatorname{cosech} x-\operatorname{cosech}(x-\mu)-\operatorname{cosech}(x+\mu)+\&c.=\frac{\pi}{\mu}\left\{\operatorname{cosec}\frac{\pi x}{\mu}-\frac{4\sin\frac{\pi x}{\mu}}{e^{\frac{\pi^2}{\mu}}+1}-\frac{4\sin\frac{3\pi x}{\mu}}{e^{\frac{3\pi^2}{\mu}}+1}-\&c.\right\}. \quad (23)$$

The comparison of the forms for $\frac{1}{\Delta \operatorname{am} u}$, (7) and (16), merely gives an equation which, on replacement of x by $x+\frac{1}{2}\mu$, is identical with that resulting from $\Delta \operatorname{am} u$, viz. (20), while the forms of $\cot \operatorname{am} u$, (8) and (17), lead at once to (21).

In the expressions on the left-hand side of (19) and (22) the number of terms included must be uneven.

It is proper to remark that the formulæ for $\phi x-\phi(x-\mu)-\phi(x+\mu)+\&c.$ can be readily deduced from those for $\phi x+\phi(x-\mu)+\phi(x+\mu)+\&c.$; thus (18) is a consequence of (20) and (23) of (21). For *ex. gr.* in (20) write 2μ for μ, and we have

$$\operatorname{sech} x+\operatorname{sech}(x-2\mu)+\operatorname{sech}(x+2\mu)+\&c.=\frac{\pi}{2\mu}\left\{1+\frac{2\cos\frac{\pi x}{\mu}}{\cosh\frac{\pi^2}{2\mu}}+\frac{2\cos\frac{2\pi x}{\mu}}{\cosh\frac{\pi^2}{\mu}}+\&c.\right\}.$$

Double this result and subtract (20) from it, and we have (18). In a similar way (23) follows from (21).

The converse proposition is not true, viz. given the value of $\phi x-\phi(x-\mu)-\phi(x+\mu)+\&c.$, we cannot deduce the value of $\phi x+\phi(x-\mu)+\phi(x+\mu)+\&c.$

§ 7. The results admit of being connected directly with FOURIER'S theorem in the following manner: it is of course well known that every integral of the form

$$\int_0^\pi \phi(x)\cos nx\,dx=A'_n,$$

or, let us write,

$$\int_0^\mu \phi(x)\cos\frac{n\pi x}{\mu}\,dx=A_n,$$

gives rise to a series

$$\phi x=\frac{1}{\mu}\left\{A_0+2A_1\cos\frac{\pi x}{\mu}+2A_2\cos\frac{2\pi x}{\mu}+\&c.\right\};$$

and that similarly from

$$\int_0^\mu \phi(x)\sin\frac{n\pi x}{\mu}\,dx=B_n$$

there follows

$$\phi x=\frac{2}{\mu}\left\{B_1\sin\frac{\pi x}{\mu}+B_2\sin\frac{2\pi x}{\mu}+\&c.\right\};$$

and it will now be shown that if ϕx is an even function of x, and if

$$\int_0^\infty \phi(x)\cos\frac{n\pi x}{\mu}\,dx=A_n,$$

then

$$\phi x+\phi(x-\mu)+\phi(x+\mu)+\phi(x-2\mu)+\phi(x+2\mu)+\&c.=\frac{2}{\mu}\left\{A_0+2A_2\cos\frac{2\pi x}{\mu}+2A_4\cos\frac{4\pi x}{\mu}+\&c.\right\}, \quad (24)$$

and

$$\phi x-\phi(x-\mu)-\phi(x+\mu)+\phi(x-2\mu)+\phi(x+2\mu)-\&c.=\frac{4}{\mu}\left\{A_1\cos\frac{\pi x}{\mu}+A_3\cos\frac{3\pi x}{\mu}+\&c.\right\}; \quad . \quad (25)$$

also, that if ϕx is an uneven function of x, and if

$$\int_0^\infty \phi(x)\sin\frac{n\pi x}{\mu}\,dx=B_n,$$

then

$$\phi x+\phi(x-\mu)+\phi(x+\mu)+\&c.=\frac{4}{\mu}\left\{B_2\sin\frac{2\pi x}{\mu}+B_4\sin\frac{4\pi x}{\mu}+\&c.\right\}, \quad . \quad . \quad . \quad (26)$$

and

$$\phi x-\phi(x-\mu)-\phi(x+\mu)+\&c.=\frac{4}{\mu}\left\{B_1\sin\frac{\pi x}{\mu}+B_3\sin\frac{3\pi x}{\mu}+\&c.\right\}. \quad . \quad . \quad . \quad (27)$$

It is sufficient to prove one of these formulæ; take (24). Since ϕx is an even function, $\phi x+\phi(x-\mu)+\phi(x+\mu)+\&c.$ (which call ψx) is a periodic function with period μ, and the right-hand side of (24) must be of the form

$$A'_0+A'_2\cos\frac{2\pi x}{\mu}+A'_4\cos\frac{4\pi x}{\mu}+\&c.$$

Now, ϕ being even,

$$2A_{2m}=\int_{-\infty}^{\infty}\phi(x)\cos\frac{2m\pi x}{\mu}\,dx$$
$$=\left\{\ldots+\int_{-\mu}^{0}+\int_{0}^{\mu}+\int_{\mu}^{2\mu}+\ldots\right\}\phi(x)\cos\frac{2m\pi x}{\mu}\,dx.$$

But

$$\int_{-\mu}^{0}\phi(x)\cos\frac{2m\pi x}{\mu}\,dx=\int_{0}^{\mu}\phi(\xi-\mu)\cos\frac{2m\pi\xi}{\mu}\,d\xi,\text{ on taking }x=\xi-\mu,$$

and

$$\int_{\mu}^{2\mu}\phi(x)\cos\frac{2m\pi x}{\mu}\,dx=\int_{0}^{\mu}\phi(\xi+\mu)\cos\frac{2m\pi\xi}{\mu}\,d\xi,\text{ on taking }x=\xi+\mu;$$

thus

$$2A_{2m}=\int_{0}^{\mu}\{\phi\xi+\phi(\xi-\mu)+\phi(\xi+\mu)+\ldots\}\cos\frac{2m\pi\xi}{\mu}\,d\xi$$
$$=\int_{0}^{\mu}\psi(x)\cos\frac{2m\pi x}{\mu}\,dx=A'_{2m}\cdot\frac{\mu}{2},$$

unless $m=0$, in which case

$$2A_0=A'_0\cdot\mu,$$

so that (24) is proved. Formula (25) may be either obtained independently by a similar treatment of the integral

$$2A_{2m+1}=\int_{-\infty}^{\infty}\phi(x)\cos\frac{(2m+1)\pi x}{\mu}\,dx,$$

or it may be deduced from (24) by writing therein 2μ for μ (remarking that by this substitution A_{2m} becomes A_m) and subtracting (24) from the double of the equation so formed. Similar processes apply to (26) and (27).

The method by which the formulæ (24) to (27) have been just obtained is the same as that by which Sir W. THOMSON (Quarterly Journal of Mathematics, t. i. p. 316) deduced the theorem

$$e^{-x^2}-e^{-(x-\mu)^2}-e^{-(x+\mu)^2}+\&c.=\frac{2\sqrt{\pi}}{\mu}\left\{e^{-\frac{\pi^2}{4\mu^2}}\cos\frac{\pi x}{\mu}+e^{-\frac{9\pi^2}{4\mu^2}}\cos\frac{3\pi x}{\mu}+\&c.\right\}\quad.\quad(28)$$

from the integral

$$\int_{0}^{\infty}e^{-x^2}\cos nx\,dx=\frac{\sqrt{\pi}}{2}e^{-\frac{n^2}{4}}.$$

It was after reading Sir W. THOMSON'S paper three or four years ago, that I made a list of all the suitable integrals of the form

$$\int_{0}^{\infty}\phi(x)\cos nx\,dx$$

that were given in Professor DE HAAN'S 'Nouvelles Tables d'Intégrales définies'

(Leyden, 1867), and deduced therefrom the resulting identities. The only formulæ so obtained which appeared of interest were, in fact, those which are given in the present paper, viz. (18) to (23); but at the time I was not aware of their connexion with the theory of Elliptic Functions. It was only recently, after obtaining the values of sin am x &c. in (10) to (17), that I remarked that the resulting identities were the same as those which I had previously deduced by the aid of Sir W. THOMSON'S principle.

It was shown by CAYLEY at the end of Sir W. THOMSON'S paper that the identity (28) corresponds to

$$\Theta(ui, k)=\sqrt{\left(\frac{\mathrm{K}}{\mathrm{K}'}\right)}e^{\frac{\pi u^2}{4\mathrm{KK}'}}\,\mathrm{H}(u+\mathrm{K}', k'); \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and it is singular that all the identities that follow from the method of this section thus appear to correspond either to elliptic or theta-function transformations. Speaking generally, the only evaluable integrals of the requisite form are derived from

$$\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx=\frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{a^2}} \text{ and } \int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}$$

(including as derivations the corresponding sine formulæ), of which the former give rise to theta-function relations, and the latter to elliptic-function relations.

§ 8. The integrals that produce the formulæ (18) to (23), and the manner in which the latter are obtained from them, deserve some attention. Thus

$$\int_0^\infty \frac{\cos nx}{e^x+e^{-x}}dx=\int_0^\infty \cos nx\,(e^{-x}-e^{-3x}+e^{-5x}-\&c.)dx$$

$$=\frac{1}{n^2+1^2}-\frac{3}{n^2+3^2}+\frac{5}{n^2+5^2}-\&c.$$

$$=\frac{\pi}{4}\operatorname{sech}\frac{n\pi}{2},$$

whereby (18) and (20) follow at once from (25) and (24).

In a similar way we can show that

$$\int_0^\infty \frac{\sin nx}{e^x-e^{-x}}dx=\frac{\pi}{4}\tanh\frac{n\pi}{2}=\frac{\pi}{4}\frac{e^{n\pi}-1}{e^{n\pi}+1};$$

but the series obtained from the direct application of this integral would not converge; and in order to deduce (21) and (23) from (26) and (27), it is necessary to express the integral in the form

$$\frac{\pi}{4}\left\{1-\frac{2}{e^{n\pi}+1}\right\},$$

and to make use of the formulæ

$$\tfrac{1}{2}\cot\tfrac{1}{2}\theta\ =\sin\theta+\sin 2\theta+\sin 3\theta+\&c.,$$

$$\tfrac{1}{2}\operatorname{cosec}\theta=\sin\theta+\sin 3\theta+\sin 5\theta+\&c.$$

This renders the process not so satisfactory from a logical point of view; but practically our knowledge that $\sin\theta+\sin 2\theta+$&c. and $\sin\theta+\sin 3\theta+$&c. are the FOURIER'S-theorem equivalents of $\frac{1}{2}\cot\frac{1}{2}\theta$ and $\frac{1}{2}\operatorname{cosec}\theta$ would be sufficient to leave no doubt of the accuracy of the formulæ so obtained.

In regard to the other two integrals required for (19) and (22), viz.

$$\int_0^\infty \tanh x\sin nx\,dx \quad\text{and}\quad \int_0^\infty \coth x\sin nx\,dx,$$

it is to be observed that, stated in this form, their values are indeterminate; for the former

$$=\int_0^\infty\left(1-\frac{2}{e^{2x}+1}\right)\sin nx\,dx,$$

and the latter

$$=\int_0^\infty\left(1+\frac{2}{e^{2x}-1}\right)\sin nx\,dx,$$

both of which involve $\cos\infty$. But in point of fact for our purpose the ∞ of the limit of the integral is not arbitrary, but is to be of the form $(m+1)\pi$, the lower limit being $-m\pi$ (or if we replace $\sin nx$ by $\sin\frac{n\pi x}{\mu}$, the limits are $(m+1)\mu$ and $-m\mu$). Taking then m infinite,

$$\int_0^{(m+1)\pi}\tanh x\sin nx\,dx=\int_0^{(m+1)\pi}\sin nx\,dx-2\int_0^\infty\frac{\sin nx}{e^{2x}+1}\,dx$$

$$=\left[-\frac{\cos nx}{n}\right]_0^{(m+1)\pi}-2\left\{\frac{n}{n^2+2^2}-\frac{n}{n^2+4^2}+\frac{n}{n^2+6^2}-\text{\&c.}\right\}$$

$$=\left[-\frac{\cos nx}{n}\right]_0^{(m+1)\pi}-\frac{n}{2}\left\{\frac{1}{(\frac{1}{2}n)^2+1^2}-\frac{1}{(\frac{1}{2}n)^2+2^2}+\frac{1}{(\frac{1}{2}n)^2+3^2}-\text{\&c.}\right\}$$

$$=\left[-\frac{\cos nx}{n}\right]_0^{(m+1)\pi}-\frac{1}{n}+\frac{\pi}{2}\operatorname{cosech}\frac{n\pi}{2}.$$

Similarly

$$\int_{-m\pi}^0\tanh x\sin nx\,dx=\int_0^{m\pi}\tanh x\sin nx\,dx$$

$$=\left[-\frac{\cos nx}{n}\right]_0^{m\pi}-\frac{1}{n}+\frac{\pi}{2}\operatorname{cosech}\frac{n\pi}{2},$$

and therefore

$$\int_{-m\pi}^{(m+1)\pi}\tanh x\sin nx\,dx=\left[-\frac{\cos nx}{n}\right]_0^{(m+1)\pi}+\left[-\frac{\cos nx}{n}\right]_0^{m\pi}-\frac{2}{n}+\pi\operatorname{cosech}\frac{n\pi}{2}$$

$$=\pi\operatorname{cosech}\frac{n\pi}{2}, \qquad (30)$$

whether m be even or uneven, if n is uneven; whence the result in (19) follows directly.

A similar course of procedure shows that

$$\int_{-m\pi}^{(m+1)\pi} \coth x \sin nx\, dx = \pi \coth \frac{n\pi}{2} = \pi\left\{1+\frac{2}{e^{n\pi}-1}\right\},$$

from which (22) may be derived.

In his 'Nouvelles Tables,' T. 265, Prof. De Haan assigns definite values to the indeterminate integrals

$$\int_0^\infty \tanh x \sin nx\, dx \text{ and } \int_0^\infty \coth x \sin nx\, dx\,;$$

and it is noticeable that, if these values be used, they lead to the same results as those just investigated. The reason is that the integrals in De Haan are in effect evaluated on the assumption that $\cos\infty=0$; and if in (30) we had, in place of the first two terms, viz.

$$\pm\frac{1}{n}+\frac{1}{n}\mp\frac{1}{n}+\frac{1}{n},$$

written

$$0+\frac{1}{n}+0+\frac{1}{n},$$

it is clear that the final result would have been the same.

It may be remarked that the identities (19) and (22) may be somewhat generalized by means of the integrals

$$\int_0^\infty \frac{\sinh ax}{\cosh bx} \sin nx\, dx = \frac{\pi}{b}\, \frac{\sinh \frac{n\pi}{2b} \sin \frac{a\pi}{2b}}{\cosh \frac{n\pi}{b} + \cos \frac{a\pi}{b}},$$

$$\int_0^\infty \frac{\cosh ax}{\sinh bx} \sin nx\, dx = \frac{\pi}{2b}\, \frac{\sinh \frac{n\pi}{b}}{\cosh \frac{n\pi}{b} + \cos \frac{a\pi}{b}}\,;$$

while other identities may be derived from

$$\int_0^\infty \frac{\cosh ax}{\cosh bx} \cos nx\, dx = \frac{\pi}{b}\, \frac{\cosh \frac{n\pi}{2b} \cos \frac{a\pi}{2b}}{\cosh \frac{n\pi}{b} + \cos \frac{a\pi}{b}},$$

$$\int_0^\infty \frac{\sinh ax}{\sinh bx} \cos nx\, dx = \frac{\pi}{2b}\, \frac{\sin \frac{a\pi}{b}}{\cosh \frac{n\pi}{b} + \cos \frac{a\pi}{a}}\,;$$

in which, of course, a is to be supposed less than b.

§ 9. The well-known reciprocity of f and ϕ in the formulæ

$$f(n)=\sqrt{\left(\frac{2}{\pi}\right)}\cdot\int_0^\infty \phi(x)\cos nx\, dx, \quad f(n)=\sqrt{\left(\frac{2}{-}\right)}\cdot\int_0^\infty \phi(x)\sin nx\, dx$$

leads to a corresponding reciprocity in the formulæ (24) to (27). Thus from the first of the integrals we deduce that, ϕ and f being both even functions, if

$$\phi x+\phi(x-\mu)+\phi(x+\mu)+\&c.=\frac{\sqrt{(2\pi)}}{\mu}\left\{f(0)+2f\left(\frac{2\pi}{\mu}\right)\cos\frac{2\pi x}{\mu}+2f\left(\frac{4\pi}{\mu}\right)\cos\frac{4\pi x}{\mu}+\&c.\right\},$$

then

$$fx+f(x-\mu)+f(x+\mu)+\&c.=\frac{\sqrt{(2\pi)}}{\mu}\left\{\phi(0)+2\phi\left(\frac{2\pi}{\mu}\right)\cos\frac{2\pi x}{\mu}+2\phi\left(\frac{4\pi}{\mu}\right)\cos\frac{4\pi x}{\mu}+\&c.\right\};$$

and if

$$\phi x-\phi(x-\mu)-\phi(x+\mu)+\&c.=\frac{2\sqrt{(2\pi)}}{\mu}\left\{f\left(\frac{\pi}{\mu}\right)\cos\frac{\pi x}{\mu}+f\left(\frac{3\pi}{\mu}\right)\cos\frac{3\pi x}{\mu}+\&c.\right\},$$

then

$$fx-f(x-\mu)-f(x+\mu)+\&c.=\frac{2\sqrt{(2\pi)}}{\mu}\left\{\phi\left(\frac{\pi}{\mu}\right)\cos\frac{\pi x}{\mu}+\phi\left(\frac{3\pi}{\mu}\right)\cos\frac{3\pi x}{\mu}+\&c.\right\}.$$

Also, from the second integral, ϕ and f being uneven, if

$$\phi x+\phi(x-\mu)+\phi(x+\mu)+\&c.=\frac{2\sqrt{(2\pi)}}{\mu}\left\{f\left(\frac{2\pi}{\mu}\right)\sin\frac{2\pi x}{\mu}+f\left(\frac{4\pi}{\mu}\right)\sin\frac{4\pi x}{\mu}+\&c.\right\},$$

then

$$fx+f(x-\mu)+f(x+\mu)+\&c.=\frac{2\sqrt{(2\pi)}}{\mu}\left\{\phi\left(\frac{2\pi}{\mu}\right)\sin\frac{2\pi x}{\mu}+\phi\left(\frac{4\pi}{\mu}\right)\sin\frac{4\pi x}{\mu}+\&c.\right\};$$

and if

$$\phi x-\phi(x-\mu)-\phi(x+\mu)+\&c.=\frac{2\sqrt{(2\pi)}}{\mu}\left\{f\left(\frac{\pi}{\mu}\right)\sin\frac{\pi x}{\mu}+f\left(\frac{3\pi}{\mu}\right)\sin\frac{3\pi x}{\mu}+\&c.\right\},$$

then

$$fx-f(x-\mu)-f(x+\mu)+\&c.=\frac{2\sqrt{(2\pi)}}{\mu}\left\{\phi\left(\frac{\pi}{\mu}\right)\sin\frac{\pi x}{\mu}+\phi\left(\frac{3\pi}{\mu}\right)\sin\frac{3\pi x}{\mu}+\&c.\right\}.$$

Applying these formulæ to the identities (18) to (23), we see that (20) is its own reciprocal, as also is the case with (18), (22), and (28); while (19) and (23) are reciprocal to one another. Although CAUCHY, in his memoir "Sur les Fonctions réciproques" (Exercices de Mathématiques, seconde année, 1827), has deduced, by means of his calculus of residues, a theorem which is in fact (24), he does not appear to have specially remarked the reciprocal character of the equations.

The application of the formulæ presents no difficulty. For example, comparing (18) with the first of the second pair, we have

$$\phi x=\text{sech } x, \qquad fx=\sqrt{\left(\frac{\pi}{2}\right)}\,.\,\text{sech}\,\frac{\pi x}{2};$$

whence the reciprocal formula is

$$\sqrt{\left(\frac{\pi}{2}\right)}\left\{\text{sech}\,\frac{\pi x}{2}-\text{sech}\,\frac{\pi(x-\mu)}{2}-\text{sech}\,\frac{\pi(x+\mu)}{2}+\&c.\right\}$$
$$=\frac{2\sqrt{(2\pi)}}{\mu}\left\{\text{sech}\,\frac{\pi}{\mu}\cos\frac{\pi x}{\mu}+\text{sech}\,\frac{3\pi}{\mu}\cos\frac{3\pi x}{\mu}+\&c.\right\},$$

which, on replacing $\frac{1}{2}\pi x$ and $\frac{1}{2}\pi\mu$ by x and μ respectively, coincides with the original formula (18)

§ 10. On looking at the formulæ (18) to (23) it appears that although we have transformations for sech $x\pm$ sech $(x-\mu)\pm$ sech $(x+\mu)+$&c., cosech $x\pm$ cosech $(x-\mu)$

$\pm$cosech$(x+\mu)+$&c., tanh $x-$tanh $(x-\mu)-$tanh$(x+\mu)+$&c., and coth $x-$coth $(x-\mu)$ $-$coth $(x+\mu)+$&c., there is none for either

$$\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c.$$

or

$$\coth x+\coth(x-\mu)+\coth(x+\mu)+\&c.;$$

it is therefore interesting to inquire what are the corresponding formulæ in these cases. If we write (21) in the form

$$\operatorname{cosech} x+\operatorname{cosech}(x-\mu)+\operatorname{cosech}(x+\mu)+\&c.=\frac{2\pi}{\mu}\left\{\tanh\frac{\pi^2}{\mu}\sin\frac{2\pi x}{\mu}+\tanh\frac{2^2}{\mu}\sin\frac{4\pi x}{\mu}+\&c.\right\},$$

and reciprocate it by the third pair of formulæ of § 9, we obtain the following result,

$$\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c.$$
$$=\frac{2\pi}{\mu}\left\{\operatorname{cosech}\frac{\pi^2}{\mu}\sin\frac{2\pi x}{\mu}+\operatorname{cosech}\frac{2\pi^2}{\mu}\sin\frac{4\pi x}{\mu}+\&c.\right\}, \quad (31)$$

which apparently ought to be the first of the two formulæ sought; but in point of fact this equation (as can be shown by actual calculation, see § 16) is not true.

It seems natural to recur to the integral (30), viz.

$$\int_{-m\pi}^{(m+1)\pi}\tanh x\sin nx\,dx=\left[-\frac{\cos nx}{n}\right]_0^{(m+1)\pi}+\left[-\frac{\cos nx}{n}\right]_0^{m\pi}-\frac{2}{n}+\pi\operatorname{cosech}\frac{n\pi}{2},$$

from which, since the first two terms of the right-hand member vanish when n is even, we have

$$\int_{-m\mu}^{(m+1)\mu}\tanh x\sin\frac{2n\pi x}{\mu}=-\frac{\mu}{n\pi}+\pi\operatorname{cosech}\frac{n\pi^2}{\mu};$$

whence ultimately, since $\frac{1}{2}\pi-\frac{1}{2}\theta=\sin\theta+\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta+\&c.$,

$$\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c.$$

$$=\frac{2x}{\mu}-1+\frac{2\pi}{\mu}\left\{\operatorname{cosech}\frac{\pi^2}{\mu}\sin\frac{2\pi x}{\mu}+\operatorname{cosech}\frac{2\pi^2}{\mu}\sin\frac{4\pi x}{\mu}+\&c.\dots\right\}; \quad (32)$$

but this result is not true either, and for the following reason:—Let

$$\psi x=\phi x-\phi(x-\mu)-\phi(x+\mu)\dots\pm\phi(x-n\mu)\pm\phi(x+n\mu),$$

and

$$\chi x=\phi x+\phi(x-\mu)+\phi(x+\mu)\dots+\phi(x-n\mu)+\phi(x+n\mu)$$

(n infinite), and suppose ϕx is an uneven function of x which $=1$, when $x=\infty$. Then

$$\psi(x+\mu)=-\psi x\pm\phi(x-n\mu)\pm\phi(x+(n+1)\mu)$$
$$=-\psi x,$$

so that ψx is periodic; but

$$\chi(x+\mu)=\chi x-\phi(x-n\mu)+\phi(x+(n+1)\mu)$$
$$=\chi x+2,$$

so that χx is not periodic. Therefore we have no right to assume that between the limits 0 and μ of x

$$\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c.$$

can be expressed in the form

$$A_1 \sin\frac{2\pi x}{\mu}+A_2 \sin\frac{4\pi x}{\mu}+A_3 \sin\frac{6\pi x}{\mu}+\&c.,$$

the true form being

$$B_1 \sin\frac{\pi x}{\mu}+B_2 \sin\frac{2\pi x}{\mu}+B_3 \sin\frac{3\pi x}{\mu}+\&c.$$

We may, however, assume that between the limits 0 and $\frac{1}{2}\mu$ of x

$$\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c.=A_1 \sin\frac{2\pi x}{\mu}+A_2 \sin\frac{4\pi x}{\mu}+\&c.;$$

and then

$$\frac{\mu}{4}A_n=\int_0^{\frac{1}{2}\mu}\{\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c.\}\sin\frac{2n\pi x}{\mu}\,dx$$
$$=\left\{\int_0^{\frac{1}{2}\mu}+\int_{-\mu}^{-\frac{1}{2}\mu}+\int_{\mu}^{\frac{3}{2}\mu}+\&c.\right\}\tanh x\sin\frac{2n\pi x}{\mu}\,dx$$
$$=\int_0^{(2m+1)\frac{\mu}{2}}\tanh x\sin\frac{2n\pi x}{\mu}\,dx$$
$$=\left[-\frac{\mu}{2n\pi}\cos\frac{2n\pi x}{\mu}\right]_0^{(2m+1)\frac{\mu}{2}}-\int_0^{\infty}\frac{2}{e^{2x}+1}\sin\frac{2n\pi x}{\mu}\,dx$$
$$=(-)^{n+1}\frac{\mu}{2n\pi}+\frac{\pi}{2}\operatorname{cosech}\frac{n\pi^2}{\mu}.$$

We thus find that between the limits 0 and $\frac{1}{2}\mu$ of x (and therefore also between the limits $-\frac{1}{2}\mu$ and $\frac{1}{2}\mu$ of x)

$$\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c.=\frac{4}{\mu}\cdot\frac{\mu}{2\pi}\left\{\sin\frac{2\pi x}{\mu}-\frac{1}{2}\sin\frac{4\pi x}{\mu}+\&c.\right\}$$
$$+\frac{4}{\mu}\cdot\frac{\pi}{2}\left\{\operatorname{cosech}\frac{\pi^2}{\mu}\sin\frac{2\pi x}{\mu}+\operatorname{cosech}\frac{2\pi^2}{\mu}\sin\frac{4\pi x}{\mu}+\&c.\right\}$$
$$=\frac{2x}{\mu}+\frac{2\pi}{\mu}\left\{\operatorname{cosech}\frac{\pi^2}{\mu}\sin\frac{2\pi x}{\mu}+\operatorname{cosech}\frac{2\pi^2}{\mu}\sin\frac{4\pi x}{\mu}+\&c.\right\}, \quad (33)$$

the terms on the left-hand side being uneven in number, and such that for every term $\tanh(x-n\mu)$ there is also a term $\tanh(x+n\mu)$.

If we write $x+\mu$ for x in this formula (33) we increase the left-hand side by $\tanh\infty+\tanh\infty$, that is by 2, while the right-hand side is increased by $\frac{2}{\mu}\cdot\mu$, that is by 2 also; while if we replace x by $x-\mu$ both sides are diminished by 2; so that (33) is true universally for all values of x, on the understanding that the left-hand side is

$$\tanh x+\{\tanh(x-\mu)+\tanh(x+\mu)\}+\{\tanh(x-2\mu)+\tanh(x+2\mu)\}+\&c.,$$

viz. that after the first term the series is to proceed by pairs of terms; so that for every term $\tanh(x\pm n\mu)$ there is also a term $\tanh(x\mp n\mu)$, and the whole number of terms included is uneven. Thus for $x=\frac{1}{2}\mu$ the series is

$$\tanh\tfrac{1}{2}\mu+\{-\tanh\tfrac{1}{2}\mu+\tanh\tfrac{3}{2}\mu\}+\{-\tanh\tfrac{3}{2}\mu+\tanh\tfrac{5}{2}\mu\}+\&c.,$$

the value of which is unity; and not

$$\{\tanh\tfrac{1}{2}\mu-\tanh\tfrac{1}{2}\mu\}+\{\tanh\tfrac{3}{2}\mu-\tanh\tfrac{3}{2}\mu\}+\&c.,$$

which is equal to zero.

If we write $x+\frac{1}{2}\mu$ for x, and suppose the terms arranged in pairs from the beginning, we find

$$\{\tanh(x+\tfrac{1}{2}\mu)+\tanh(x-\tfrac{1}{2}\mu)\}+\{\tanh(x+\tfrac{3}{2}\mu)+\tanh(x-\tfrac{3}{2}\mu)\}+\&c.$$

$$=\frac{2x}{\mu}-\frac{2\pi}{\mu}\left\{\operatorname{cosech}\frac{\pi^2}{\mu}\sin\frac{2\pi x}{\mu}-\operatorname{cosech}\frac{2\pi^2}{\mu}\sin\frac{4\pi x}{\mu}+\&c.\right\} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

as the unity which is introduced on the right-hand side by the change is cancelled by the unity on the left-hand side, which results from the supposition that the number of terms is even.

The last equation is, in fact, the relation

$$i\mathrm{Z}(ui+\mathrm{K})=\frac{\pi u}{2\mathrm{KK}'}+\mathrm{Z}(u+\mathrm{K}',k') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

(Fundamenta Nova, p. 165, and DURÈGE, § 69); for

$$\mathrm{Z}(u)=\frac{2\pi}{\mathrm{K}}\left\{\frac{q}{1-q^2}\sin 2x+\frac{q^2}{1-q^4}\sin 4x+\frac{q^3}{1-q^6}\sin 6x+\&c.\right\};$$

so that (35) becomes

$$\frac{2\pi i}{\mathrm{K}}\left\{-\frac{q}{1-q^2}\sin 2xi+\frac{q^2}{1-q^4}\sin 4xi-\frac{q^3}{1-q^6}\sin 6xi+\&c.\right\}$$

$$=\frac{\pi u}{2\mathrm{KK}'}+\frac{2\pi}{\mathrm{K}'}\left\{-\frac{r}{1-r^2}\sin 2z+\frac{r^2}{1-r^4}\sin 4z-\&c.\right\},$$

of which the left-hand side

$$=\frac{\pi}{\mathrm{K}}\{(e^{2x}-e^{-2x})(q+q^3+q^5+\&c.)-(e^{4x}-e^{-4x})(q^2+q^6+q^{10}+\&c.)+\&c.\}$$

$$=\frac{\pi}{\mathrm{K}}\left\{\frac{qe^{2x}}{1+qe^{2x}}-\frac{qe^{-2x}}{1+qe^{-2x}}+\frac{q^3e^{2x}}{1+q^3e^{2x}}-\frac{q^3e^{-2x}}{1+q^3e^{-2x}}+\&c.\right\}$$

$$=-\frac{\pi}{2\mathrm{K}}\left\{\frac{1-qe^{2x}}{1+qe^{2x}}-\frac{1-qe^{-2x}}{1+qe^{-2x}}+\frac{1-q^3e^{2x}}{1+q^3e^{2x}}-\frac{1-q^3e^{-2x}}{1+q^3e^{-2x}}+\&c.\right\};$$

and the identity becomes

$$\tanh(x-\tfrac{1}{2}\mu)+\tanh(x+\tfrac{1}{2}\mu)+\tanh(x-\tfrac{3}{2}\mu)+\tanh(x+\tfrac{3}{2}\mu)+\&c.$$

$$=\frac{2x}{\mu}-\frac{2\pi}{\mu}\left\{\frac{\sin 2z}{\sinh\nu}-\frac{\sin 4z}{\sinh 2\nu}+\&c.\right\},$$

z being $\frac{\pi x}{\mu}$ and ν being $\frac{\pi^2}{\mu}$. We see from this investigation also that the left-hand side must consist of an even number of pairs of terms.

As (35) is obtained by differentiating logarithmically the formula

$$\Theta(ui+\mathrm{K})=\sqrt{\left(\frac{\mathrm{K}}{\mathrm{K}'}\right)}e^{\frac{\pi u^2}{4\mathrm{KK}'}}\Theta(u+\mathrm{K}', k'),$$

it follows that (34) is a form of the identity that results from differentiating logarithmically

$$e^{-x^2}+e^{-(x-\mu)^2}+e^{-(x+\mu)^2}+\&c.=\frac{\sqrt{\pi}}{\mu}\left\{1+2e^{-\frac{\pi^2}{\mu^2}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{4\pi^2}{\mu^2}}\cos\frac{4\pi x}{\mu}+\&c.\right\}.$$

The formula corresponding to (33) for the hyperbolic cotangent can be shown, by a process similar to that by which (33) was itself established, to be

$$\coth x+\coth(x-\mu)+\coth(x+\mu)+\&c.=\frac{2x}{\mu}+\frac{\pi}{\mu}\left\{\cot\frac{\pi x}{\mu}+\frac{4\sin\frac{2\pi x}{\mu}}{e^{\frac{2\pi^2}{\mu}}-1}+\frac{4\sin\frac{4\pi x}{\mu}}{e^{\frac{4\pi^2}{\mu}}-1}+\&c.\right\}, \quad (36)$$

which holds good universally, on the same understanding, with regard to the number and order of the terms, as that which was found requisite for the truth of (33).

§ 11. I now proceed to show how the identities which have been obtained in the preceding sections by elliptic functions, or by FOURIER's theorem, can be deduced from the ordinary formulæ for the cotangent and cosecant, viz.

$$\cot x=\frac{1}{x}+\frac{1}{x-\pi}+\frac{1}{x+\pi}+\frac{1}{x-2\pi}+\frac{1}{x+2\pi}+\&c., \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$\operatorname{cosec} x=\frac{1}{x}-\frac{1}{x-\pi}-\frac{1}{x+\pi}+\frac{1}{x-2\pi}+\frac{1}{x+2\pi}-\&c., \quad . \quad . \quad . \quad . \quad . \quad (38)$$

by elementary algebra and trigonometry.

Thus to prove (18) we have

$$\operatorname{cosec}(x+ai)=\frac{1}{x+ai}-\frac{1}{x+ai-\pi}-\frac{1}{x+ai+\pi}+\&c.,$$

$$\operatorname{cosec}(x-ai)=\frac{1}{x+ai}-\frac{1}{x-ai-\pi}-\frac{1}{x-ai+\pi}+\&c.;$$

whence, by subtraction,

$$\frac{a}{x^2+a^2}-\frac{a}{(x-\pi)^2+a^2}-\frac{a}{(x+\pi)^2+a^2}+\&c.=-\frac{1}{2i}\{\operatorname{cosec}(x+ai)-\operatorname{cosec}(x-ai)\}.$$

Now

$$\operatorname{cosec} u=\frac{2i}{e^{ui}-e^{-ui}}=\frac{2i\,e^{ui}}{e^{2ui}-1}$$

$$=-2ie^{ui}(1+e^{2ui}+e^{4ui}+\&c.);$$

whence

$$-\frac{1}{2i}\{\operatorname{cosec}(x+ai)-\operatorname{cosec}(x-ai)\}=e^{xi-a}+e^{3xi-3a}+e^{5xi-5a}+\&c.$$

$$+e^{-xi-a}+e^{-3xi-3a}+e^{-5xi-5a}+\&c.$$

$$=2\{e^{-a}\cos x+e^{-3a}\cos 3x+\&c.\};$$

and, on replacing x and a by $\frac{\pi x}{\mu}$ and $\frac{\pi a}{\mu}$, we obtain the formula

$$\frac{a}{x^2+a^2}-\frac{a}{(x-\mu)^2+a^2}-\frac{a}{(x+\mu)^2+a^2}+\frac{a}{(x-2\mu)^2+a^2}+\frac{a}{(x+2\mu)^2+a^2}-\&c.$$

$$=\frac{2\pi}{\mu}\left(e^{-\frac{\pi a}{\mu}}\cos\frac{\pi x}{\mu}+e^{-\frac{3\pi a}{\mu}}\cos\frac{3\pi x}{\mu}+\&c.\right). \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Now from (38)

$$-\sec x=\frac{1}{x-\frac{1}{2}\pi}-\frac{1}{x-\frac{3}{2}\pi}-\frac{1}{x+\frac{1}{2}\pi}+\frac{1}{x-\frac{5}{2}\pi}+\&c.$$

$$=\frac{\pi}{x^2-(\frac{1}{2}\pi)^2}-\frac{3\pi}{x^2-(\frac{3}{2}\pi)^2}+\frac{5\pi}{x^2-(\frac{5}{2}\pi)^2}-\&c.;$$

whence, writing xi for x,

$$\operatorname{sech} x=\frac{\pi}{x^2+(\frac{1}{2}\pi)^2}-\frac{3\pi}{x^2+(\frac{3}{2}\pi)^2}+\frac{5\pi}{x^2+(\frac{5}{2}\pi)^2}-\&c.,$$

and

$$-\operatorname{sech}(x-\mu)=-\frac{\pi}{(x-\mu)^2+(\frac{1}{2}\pi)^2}+\frac{3\pi}{(x-\mu)^2+(\frac{3}{2}\pi)^2}-\frac{5\pi}{(x-\mu)^2+(\frac{5}{2}\pi)^2}+\&c.$$

$$-\operatorname{sech}(x+\mu)=-\frac{\pi}{(x+\mu)^2+(\frac{1}{2}\pi)^2}+\frac{3\pi}{(x+\mu)^2+(\frac{3}{2}\pi)^2}-\frac{5\pi}{(x+\mu)^2+(\frac{5}{2}\pi)^2}+\&c.$$

$$+\operatorname{sech}(x-2\mu)=\frac{\pi}{(x-2\mu)^2+(\frac{1}{2}\pi)^2}-\frac{3\pi}{(x-2\mu)^2+(\frac{3}{2}\pi)^2}+\frac{5\pi}{(x-2\mu)^2+(\frac{5}{2}\pi)^2}-\&c.$$

. . . .

Adding these expressions together in columns, and transforming each column by (39), we find

$$\operatorname{sech} x-\operatorname{sech}(x-\mu)-\operatorname{sech}(x+\mu)+\operatorname{sech}(x-2\mu)+\operatorname{sech}(x+2\mu)-\&c.$$

$$=\frac{4\pi}{\mu}\left(e^{-\frac{\pi^2}{2\mu}}\cos\frac{\pi x}{\mu}+e^{-\frac{3\pi^2}{2\mu}}\cos\frac{3\pi x}{\mu}+e^{-\frac{5\pi^2}{2\mu}}\cos\frac{5\pi x}{\mu}+\&c.\right)$$

$$-\frac{4\pi}{\mu}\left(e^{-\frac{3\pi^2}{2\mu}}\cos\frac{\pi x}{\mu}+e^{-\frac{9\pi^2}{2\mu}}\cos\frac{3\pi x}{\mu}+e^{-\frac{15\pi^2}{2\mu}}\cos\frac{5\pi x}{\mu}+\&c.\right)$$

$$+\frac{4\pi}{\mu}\left(e^{-\frac{5\pi^2}{2\mu}}\cos\frac{\pi x}{\mu}+e^{-\frac{15\pi^2}{2\mu}}\cos\frac{3\pi x}{\mu}+e^{-\frac{25\pi^2}{2\mu}}\cos\frac{5\pi x}{\mu}+\&c.\right)$$

. . . . , which, after summation of the columns,

$$=\frac{4\pi}{\mu}\left(\frac{e^{-\frac{\pi^2}{2\mu}}}{1+e^{-\frac{\pi^2}{\mu}}}\cos\frac{\pi x}{\mu}+\frac{e^{-\frac{3\pi^2}{2\mu}}}{1+e^{-\frac{3\pi^2}{\mu}}}\cos\frac{3\pi x}{\mu}+\frac{e^{-\frac{5\pi^2}{2\mu}}}{1+e^{-\frac{5\pi^2}{\mu}}}\cos\frac{5\pi x}{\mu}+\&c.\right)$$

$$=\frac{2\pi}{\mu}\left(\operatorname{sech}\frac{\pi^2}{2\mu}\cos\frac{\pi x}{\mu}+\operatorname{sech}\frac{3\pi^2}{2\mu}\cos\frac{3\pi x}{\mu}+\operatorname{sech}\frac{5\pi^2}{2\mu}\cos\frac{5\pi x}{\mu}+\&c.\right),$$

which is the identity (18), that was in § 6 deduced from the formula

$$\cos\operatorname{am} u=\sec\operatorname{am}(ui, k'),$$

and in § 8 from the integral

$$\int_0^\infty \operatorname{sech} x\cos nx\,dx=\frac{\pi}{2}\operatorname{sech}\frac{n\pi}{2}.$$

§ 12. The other identities, (19) to (23), admit of being demonstrated in exactly the same way. The formulæ of transformation, similar to (39), that are required are

$$\frac{x}{x^2+a^2}-\frac{x-\mu}{(x-\mu)^2+a^2}-\frac{x+\mu}{(x+\mu)^2+a^2}+\&c.=\frac{2\pi}{\mu}\left(e^{-\frac{\pi a}{\mu}}\sin\frac{\pi x}{\mu}+e^{-\frac{3\pi a}{\mu}}\sin\frac{3\pi x}{\mu}+\&c.\right),$$

$$\frac{a}{x^2+a^2}+\frac{a}{(x-\mu)^2+a^2}+\frac{a}{(x+\mu)^2+a^2}+\&c.=\frac{\pi}{\mu}\left(1+2e^{-\frac{2\pi a}{\mu}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{4\pi a}{\mu}}\cos\frac{4\pi x}{\mu}+\&c.\right),$$

$$\frac{x}{x^2+a^2}+\frac{x-\mu}{(x-\mu)^2+a^2}+\frac{x+\mu}{(x+\mu)^2+a^2}+\&c.=\frac{2\pi}{\mu}\left(e^{-\frac{2\pi a}{\mu}}\sin\frac{2\pi x}{\mu}+e^{-\frac{4\pi a}{\mu}}\sin\frac{4\pi x}{\mu}+\&c.\right),$$

the first resulting from $\operatorname{cosec}(x+ai)+\operatorname{cosec}(x-ai)$, and the other two from $\cot(x+ai)\mp\cot(x-ai)$. The following expressions, which are analogous to that used for sech x in the last section, are also needed :—

$$\tanh x=\frac{2x}{x^2+(\frac{1}{2}\pi)^2}+\frac{2x}{x^2+(\frac{3}{2}\pi)^2}+\frac{2x}{x^2+(\frac{5}{2}\pi)^2}+\&c.,$$

$$\coth x=\frac{1}{x}+\frac{2x}{x^2+\pi^2}+\frac{2x}{x^2+(2\pi)^2}+\frac{2x}{x^2+(3\pi)^2}+\&c.,$$

$$\operatorname{cosech} x=\frac{1}{x}-\frac{2x}{x^2+\pi^2}+\frac{2x}{x^2+(2\pi)^2}-\frac{2x}{x^2+(3\pi)^2}+\&c.,$$

all of which follow from (37) and (38) at once in the same way as that by which the formula for sech x was obtained.

Only one point calls for notice in these demonstrations, viz. in the proof of (20) we find

$$\operatorname{sech} x+\operatorname{sech}(x-\mu)+\operatorname{sech}(x+\mu)+\&c.$$

$$=\frac{2\pi}{\mu}\left(1+2e^{-\frac{\pi^2}{\mu}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{2\pi^2}{\mu}}\cos\frac{4\pi x}{\mu}+\&c.\right)$$

$$-\frac{2\pi}{\mu}\left(1+2e^{-\frac{3\pi^2}{\mu}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{6\pi^2}{\mu}}\cos\frac{4\pi x}{\mu}+\&c.\right)$$

$$+\frac{2\pi}{\mu}\left(1+2e^{-\frac{5\pi^2}{\mu}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{10\pi^2}{\mu}}\cos\frac{4\pi x}{\mu}+\&c.\right)$$

$$-\ \ldots\ldots\ldots\ldots\ldots;$$

and in order to obtain the correct result we must replace the indeterminate series, $1-1+1-1+1-\&c.$, by $\frac{1}{2}$. Cases in which the method gives results absolutely erroneous will be noticed in § 16.

It will have been seen that the process of § 11 consists in replacing each term of the original series by n terms (n infinite), and therefore the original expression itself by n^2 terms. Each series of n terms formed by adding the vertical columns is transformed into another series of n terms, so that we thus replace the first scheme of n^2 terms by a second scheme of n^2 terms, which latter system, being such that the columns admit of being summed as ordinary geometrical progressions, gives the second side of the identity to be proved.

§ 13. A question that naturally arises is to inquire what are the results which we should obtain if, instead of using (39) and the similar formulæ for the conversion of one series into another, we were to replace at once these series by their finite summations, *i. e.* instead of (39) to take

$$\frac{a}{x^2+a^2}-\frac{a}{(x-\mu)^2+a^2}-\frac{a}{(x+\mu)^2+a^2}+\&c.=\frac{\pi}{2\mu i}\left\{\operatorname{cosec}\frac{\pi}{\mu}(x-ai)-\operatorname{cosec}\frac{\pi}{\mu}(x+ai)\right\}$$

$$=\frac{\pi}{2\mu i}\,\frac{2\cos\frac{\pi x}{\mu}\sin\frac{\pi a i}{\mu}}{\sin^2\frac{\pi x}{\mu}+\sin^2\frac{\pi a i}{\mu}}$$

$$=\frac{\pi}{\mu}\,\frac{\cos\frac{\pi x}{\mu}\sinh\frac{\pi a}{\mu}}{\sin^2\frac{\pi x}{\mu}+\sinh^2\frac{\pi a}{\mu}}.$$

We thus find

$$\operatorname{sech} x-\operatorname{sech}(x-\mu)-\operatorname{sech}(x+\mu)+\&c.$$

$$=\frac{2\pi}{\mu}\cos\frac{\pi x}{\mu}\left\{\frac{\sinh\frac{\pi^2}{2\mu}}{\sin^2\frac{\pi x}{\mu}+\sinh^2\frac{\pi^2}{2\mu}}-\frac{\sinh\frac{3\pi^2}{2\mu}}{\sin^2\frac{\pi x}{\mu}+\sinh^2\frac{3\pi^2}{2\mu}}+\&c.\right\}, \quad . \quad . \quad . \quad . \quad (40)$$

while the left-hand side also

$$=\frac{2\pi}{\mu}\left(\operatorname{sech}\frac{\pi^2}{2\mu}\cos\frac{\pi x}{\mu}+\operatorname{sech}\frac{3\pi^2}{2\mu}\cos\frac{3\pi x}{\mu}+\&c.\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

from (18). Although (40) is the identity which we have absolutely proved, we may regard the fresh identity as being that which follows from (40) and (41), viz. (writing for the moment x in place of $\frac{\pi x}{\mu}$, and μ in place of $\frac{\pi^2}{\mu}$)

$$\frac{\cos x}{\cosh\frac{1}{2}\mu}+\frac{\cos 3x}{\cosh\frac{3}{2}\mu}+\&c.=\cos x\left\{\frac{\sinh\frac{1}{2}\mu}{\sin^2 x+\sinh^2\frac{1}{2}\mu}-\frac{\sinh\frac{3}{2}\mu}{\sin^2 x+\sinh^2\frac{3}{2}\mu}+\&c.\right\}. \quad . \quad . \quad (42)$$

This result follows immediately from another form of the series for the cosine amplitude; for on p. 113 of his 'Lehre von den elliptischen Integralen und den Theta-Functionen' (Berlin, 1864), SCHELLBACH finds

$$\theta o\, \theta_2 o\, gx = 4 \cos x\, \Sigma_0^\infty \frac{(-)^s q^{s+\frac{1}{2}}(1-q^{2s+1})}{1-2q^{2s+1}\cos 2x + q^{4s+2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

We easily see that

$$\theta o\, \theta_2 o\, gx = \frac{2k\mathrm{K}}{\pi} \cos \mathrm{am} \frac{2\mathrm{K}x}{\pi} = 4\left\{\frac{q^{\frac{1}{2}}}{1+q}\cos x + \frac{q^{\frac{3}{2}}}{1+q^3}\cos 3x + \&\mathrm{c.}\right\}, \quad . \quad . \quad . \quad (44)$$

and the comparison of (43) and (44) at once gives (42), since $\sin^2 x + \sinh^2 a = \frac{1}{2}(\cosh 2a - \cos 2x)$. The result (43) is also given in the 'Fundamenta Nova,' p. 102.

It thus appears that by absolutely summing, instead of transforming, in the process of § 11 we obtain the series of formulæ which SCHELLBACH has given on pp. 113, 114 of his treatise, so that all the formulæ and identities which arise from the transformations of the elliptic functions are algebraically exhibited by the method of § 11. It is unnecessary to write down the series of identities analogous to (42) for the other functions, as they can be easily derived as above from the values in SCHELLBACH. It may be remarked that (40) is a transformation of sec am $(ui, k') = \cos \mathrm{am}\, u$, but (42) is merely a transformation of cos am $u = \cos \mathrm{am}\, u$. If, therefore, we perform the process of § 11 in reverse order (*i. e.* starting with the trigonometrical side of the identity to be proved, sum the rows instead of transforming them) we obtain (42) at once.

It appears at first sight as if SCHELLBACH'S formula

$$\frac{2k'\mathrm{K}}{\pi} \sec \mathrm{am} \frac{2\mathrm{K}x}{\pi} = \sec x + 4 \cos x\, \Sigma_1^\infty \frac{(-)^s q^s (1+q^{2s})}{1+2q^{2s}\cos 2x + q^{4s}} \quad . \quad . \quad . \quad . \quad (45)$$

gave rise to another formula for the cosine amplitude, by writing xi for x and changing the modulus from k to k'; but this, in fact, merely gives an expression already obtained; for the right-hand side of (45), on writing xi for x and $e^{-\mu}$ for q, becomes

$$\mathrm{sech}\, x + 4 \cosh x\, \Sigma_1^\infty \frac{(-)^s \cosh s\mu}{\cosh 2x + \cosh 2s\mu},$$

which

$$= \mathrm{sech}\, x + \Sigma_1^\infty (-)^s \frac{\cosh(x-s\mu) + \cosh(x+s\mu)}{\cosh(x-s\mu)\cosh(x+s\mu)}$$

$$= \mathrm{sech}\, x + \Sigma_1^\infty (-)^s \{\mathrm{sech}(x-s\mu) + \mathrm{sech}(x+s\mu)\}.$$

Formulæ such as (45) are the nearest approach I have met with to those numbered (10) to (17) and the other expressions at the end of § 5; but (besides that an imaginary transformation is required to reduce them to these forms) they do not put in evidence the periodicity of the functions.

§ 14. It is perhaps desirable to place side by side, for convenience of comparison, all the different forms into which one of the functions, the cosine amplitude, has now been thrown. Writing, as before,

$$x = \frac{\pi u}{2\mathrm{K}}, \quad z = \frac{\pi u}{2\mathrm{K}'}, \quad q = e^{-\frac{\pi \mathrm{K}'}{\mathrm{K}}} = e^{-\mu}, \quad r = e^{-\frac{\pi \mathrm{K}}{\mathrm{K}'}} = e^{-\nu},$$

$$\cos \operatorname{am} u = \frac{2\pi}{k\text{K}}\left\{\frac{q^{\frac{1}{2}}}{1+q}\cos x + \frac{q^{\frac{3}{2}}}{1+q^3}\cos 3x + \frac{q^{\frac{5}{2}}}{1+q^5}\cos 5x + \&\text{c.}\right\}$$

$$= \frac{2\pi}{k\text{K}}\cos x\left\{\frac{q^{\frac{1}{2}}(1-q)}{1-2q\cos 2x+q^2} - \frac{q^{\frac{3}{2}}(1-q^3)}{1-2q^3\cos 2x+q^6} + \&\text{c.}\right\}$$

$$= \frac{\pi}{k\text{K}'}\left\{\frac{1}{r^{\frac{x}{\pi}}+r^{-\frac{x}{\pi}}} - \frac{1}{r^{\frac{x}{\pi}-1}+r^{-\left(\frac{x}{\pi}-1\right)}} - \frac{1}{r^{\frac{x}{\pi}+1}+r^{-\left(\frac{x}{\pi}+1\right)}} + \&\text{c.}\right\}$$

$$= \frac{\pi}{2k\text{K}'}\{\operatorname{sech} z - \operatorname{sech}(z-\nu) - \operatorname{sech}(z+\nu) + \&\text{c.}\}$$

$$= \frac{\pi}{2k\text{K}'}\left\{\operatorname{sech} z - 4\cosh z\left(\frac{\cosh \nu}{\cosh 2z+\cosh 2\nu} - \frac{\cosh 2\nu}{\cosh 2z+\cosh 4\nu} + \&\text{c.}\right)\right\}$$

$$= \frac{\pi}{2k\text{K}'}\left\{\operatorname{sech} z - \frac{4r}{1+r}\cosh z + \frac{4r^3}{1+r^3}\cosh 3z - \&\text{c.}\right\};$$

while x, z, μ, ν being any four quantities subject to the relations

$$\mu\nu = \pi^2, \quad z = \frac{\pi x}{\mu}\left(\text{whence } x = \frac{\pi z}{\nu}\right),$$

the identities are:—

$$\operatorname{sech} x - \operatorname{sech}(x-\mu) - \operatorname{sech}(x+\mu) + \operatorname{sech}(x-2\mu) + \operatorname{sech}(x+2\mu) - \&\text{c.}$$

$$= \operatorname{sech} x - 4\cosh x\left\{\frac{\cosh \mu}{\cosh 2x+\cosh 2\mu} - \frac{\cosh 2\mu}{\cosh 2x+\cosh 4\mu} + \&\text{c.}\right\}$$

$$= \operatorname{sech} x - \frac{4\cosh x}{e^{\mu}+1} + \frac{4\cosh 3x}{e^{3\mu}+1} - \&\text{c.}$$

$$= \frac{2\pi}{\mu}\left\{\frac{\cosh z}{\cosh \frac{1}{2}\nu} + \frac{\cosh 3z}{\cosh \frac{3}{2}\nu} + \frac{\cosh 5z}{\cosh \frac{5}{2}\nu} + \&\text{c.}\right\}$$

$$= \frac{2\pi}{\mu}\cos z\left\{\frac{\sinh \frac{1}{2}\nu}{\sin^2 z+\sinh^2 \frac{1}{2}\nu} - \frac{\sinh \frac{3}{2}\nu}{\sin^2 z+\sinh^2 \frac{3}{2}\nu} + \&\text{c.}\right\}.$$

Another form will also be given in the next section. It is scarcely necessary to observe that corresponding formulæ and identities exist for $\sin \operatorname{am} u$, $\Delta \operatorname{am} u$, $\operatorname{cosec} \operatorname{am} u$, $\frac{\sin \operatorname{am} u}{\Delta \operatorname{am} u}$, &c.

§ 15. The identities (18) to (23) can also be proved by trigonometry in another distinct manner, by starting from the trigonometrical sides of the equations. Thus, for (18), from the formula

$$\frac{\pi}{4}\operatorname{sech} \tfrac{1}{2}\pi\beta = \frac{1}{1^2+\beta^2} - \frac{3}{3^2+\beta^2} + \frac{5}{5^2+\beta^2} - \&\text{c.},$$

we have (writing z for $\frac{\pi x}{\mu}$ for brevity)

$$\frac{\pi}{4}\operatorname{sech}\frac{\pi^2}{2\mu}\cos z = \frac{\mu^2}{\pi^2}\left\{\frac{\cos z}{\frac{\mu^2}{\pi^2}+1^2} - \frac{3\cos z}{\frac{3^2\mu^2}{\pi^2}+1^2} + \frac{5\cos z}{\frac{5^2\mu^2}{\pi^2}+1^2} - \&\text{c.}\right\}$$

$$\frac{\pi}{4}\operatorname{sech}\frac{3\pi^2}{2\mu}\cos 3z=\frac{\mu^2}{\pi^2}\left\{\frac{\cos 3z}{\frac{\mu^2}{\pi^2}+3^2}-\frac{3\cos 3z}{\frac{3^2\mu^2}{\pi^2}+3^2}+\frac{5\cos 3z}{\frac{5^2\mu^2}{\pi^2}+3^2}-\&c.\right\}$$

$$\frac{\pi}{4}\operatorname{sech}\frac{5\pi^2}{2\mu}\cos 5z=\frac{\mu^2}{\pi^2}\left\{\frac{\cos 5z}{\frac{\mu^2}{\pi^2}+5^2}-\frac{3\cos 5z}{\frac{3^2\mu^2}{\pi^2}+5^2}+\frac{5\cos 5z}{\frac{5^2\mu^2}{\pi^2}+5^2}-\&c.\right\}$$

. . . .;

whence

$$\frac{2\pi}{\mu}\left\{\operatorname{sech}\frac{\pi^2}{2\mu}\cos z+\operatorname{sech}\frac{3\pi^2}{2\mu}\cos 3z+\operatorname{sech}\frac{5\pi^2}{2\mu}\cos 5z+\&c.\right\}$$

$$=\frac{8\mu}{\pi^2}\left\{\frac{\cos z}{\frac{\mu^2}{\pi^2}+1^2}+\frac{\cos 3z}{\frac{\mu^2}{\pi^2}+3^2}+\frac{\cos 5z}{\frac{\mu^2}{\pi^2}+5^2}+\&c.\right\}$$

$$-\frac{8\mu}{\pi^2}\left\{\frac{3\cos z}{\frac{3^2\mu^2}{\pi^2}+1^2}+\frac{3\cos 3z}{\frac{3^2\mu^2}{\pi^2}+3^2}+\frac{3\cos 5z}{\frac{3^2\mu^2}{\pi^2}+5^2}+\&c.\right\}$$

$$+\ \ldots\ldots\ldots\ldots$$

$$=2\left\{\frac{\sinh(\tfrac{1}{2}\mu-x)}{\cosh\tfrac{1}{2}\mu}-\frac{\sinh 3(\tfrac{1}{2}\mu-x)}{\cosh\tfrac{3}{2}\mu}+\frac{\sinh 5(\tfrac{1}{2}\mu-x)}{\cosh\tfrac{5}{2}\mu}-\&c.\right\}$$

$$=2\left\{\left(e^{-x}-\frac{2\cosh x}{1+e^{\mu}}\right)-\left(e^{-3x}-\frac{2\cosh 3x}{1+e^{3\mu}}\right)+\&c.\right\}$$

$$=\frac{2}{e^{x}+e^{-x}}-\frac{4\cosh x}{1+e^{\mu}}+\frac{4\cosh 3x}{1+e^{3\mu}}-\&c.,$$

which, as shown in § 4,

$$=\operatorname{sech}x-\operatorname{sech}(x-\mu)-\operatorname{sech}(x+\mu)+\&c.$$

We thus in the course of the proof obtain another form for $\operatorname{sech}x-\operatorname{sech}(x-\mu)-\operatorname{sech}(x+\mu)+\&c.$, viz.

$$2\left\{\frac{\sinh(\tfrac{1}{2}\mu-x)}{\cosh\tfrac{1}{2}\mu}-\frac{\sinh 3(\tfrac{1}{2}\mu-x)}{\cosh\tfrac{3}{2}\mu}+\frac{\sinh 5(\tfrac{1}{2}\mu-x)}{\cosh\tfrac{5}{2}\mu}-\&c.\right\};\quad\ldots\ldots\quad(46)$$

whence, in addition to the forms for cos am u in § 14, we have

$$\cos\operatorname{am}u=\frac{\pi}{k\mathrm{K}'}\left\{\frac{\sinh(\tfrac{1}{2}\nu-z)}{\cosh\tfrac{1}{2}\nu}-\frac{\sinh 3(\tfrac{1}{2}\nu-z)}{\cosh\tfrac{3}{2}\nu}+\&c.\right\}.$$

This method of proof is not so interesting as that of § 11, both because the formulæ required cannot be obtained in so elementary a manner, and also because the identities (18) to (23) are not so directly verified, as their right-hand members are shown to be equal to expressions such as (46), which themselves need some transformation before they assume the desired forms. The formula

$$\frac{\cos x}{1^2+\beta^2}+\frac{\cos 3x}{3^2+\beta^2}+\&c.=\frac{\pi}{4\beta}\,\frac{\sinh(\tfrac{1}{2}\beta\pi-\beta x)}{\cosh\tfrac{1}{2}\beta\pi},$$

which was required in the verification, is best obtained by deducing it from the well-known theorem

$$\frac{\cos x}{1^2+\beta^2}+\frac{\cos 2x}{2^2+\beta^2}+\frac{\cos 3x}{3^2+\beta^2}+\&c.=\frac{\pi}{2\beta}\frac{\cosh(\beta\pi-\beta x)}{\sinh\beta\pi}-\frac{1}{2\beta^2}, \quad . \quad . \quad . \quad . \quad (47)$$

from which, by writing $\frac{1}{2}\beta$ for β and $2x$ for x, dividing the equation so obtained by 4, and subtracting it from (47), we find

$$\frac{\cos x}{1^2+\beta^2}+\frac{\cos 3x}{3^2+\beta^2}+\&c.=\frac{\pi}{2\beta}\left\{\frac{\cosh(\beta\pi-\beta x)}{\sinh\beta\pi}-\frac{1}{2}\frac{\cosh(\frac{1}{2}\beta\pi-\beta x)}{\sinh\frac{1}{2}\beta\pi}\right\}$$

$$=\frac{\pi}{2\beta}\left\{\frac{\cosh(\beta\pi-\beta x)-\cosh(\frac{1}{2}\beta\pi-\beta x)\cosh\frac{1}{2}\beta\pi}{\sinh\beta\pi}\right\}$$

$$=\frac{\pi}{4\beta}\cdot\frac{\sinh(\frac{1}{2}\beta\pi-\beta x)}{\cosh\frac{1}{2}\beta\pi}.$$

It is to be noticed that (46) is only true if x lies between 0 and μ. This may be regarded as a consequence of the fact that (47) only holds good when x is positive and less than 2π; but the necessity for the condition is also evident from the process of verification by ordinary algebra. Thus the expression in (46)

$$=2\left\{e^{-x}-e^{-3x}+e^{-5x}-\ldots-\frac{2\cosh x}{1+e^{\mu}}+\frac{2\cosh 3x}{1+e^{3\mu}}-\&c.\right\}$$

$$=\frac{2e^{-x}}{1+e^{-2x}}-2(e^{x}+e^{-x})(e^{-\mu}-e^{-2\mu}+e^{-3\mu}-\&c.)+2(e^{3x}+e^{-3x})(e^{-3\mu}-e^{-6\mu}+e^{-9\mu}-\ldots)-\&c.$$

$$=\frac{2}{e^{x}+e^{-x}}-\frac{2e^{x-\mu}}{1+e^{2(x-\mu)}}-\frac{2e^{x+\mu}}{1+e^{2(x+\mu)}}+\&c.$$

$$=\operatorname{sech}x-\operatorname{sech}(x-\mu)-\operatorname{sech}(x+\mu)+\&c.,$$

wherein we see that to justify the summations of $e^{-x}-e^{-3x}+\&c$, and $e^{x-\mu}-e^{3(x-\mu)}+\&c.$ as ordinary geometrical progressions we must suppose x to be positive and less than μ. Also since sech $x-$sech $(x-\mu)-$sech$(x+\mu)+\&c.$ is periodic, while the expression in (46) is not so, we see that the equality will not hold good beyond these limits.

I have worked out the corresponding proofs of the other five identities (19) to (23) in the same way, but none of them call for any special remark. The process is not in all cases exactly similar, as, *ex. gr.*, in deducing (19) from

$$\frac{\pi}{\sinh\beta\pi}=\frac{1}{\beta}-\frac{2\beta}{\beta^2+1}+\frac{2\beta}{\beta^2+2^2}-\&c.,$$

$$\frac{\sin x}{\beta^2+1^2}+\frac{3\sin 3x}{\beta^2+3^2}+\frac{5\sin 5x}{\beta^2+5^2}+\&c.=\frac{\pi}{4}\frac{\cosh(\frac{1}{2}\beta\pi-\beta x)}{\cosh\frac{1}{2}\beta\pi},$$

we find

$$\frac{\sin z}{\sinh\frac{1}{2}\nu}+\frac{\sin 3z}{\sinh\frac{3}{2}\nu}+\&c.=\frac{2\mu}{\pi^2}(\sin z+\tfrac{1}{3}\sin 3z+\tfrac{1}{5}\sin 5z+\&c.)$$

$$-\frac{\mu}{\pi}\left\{\frac{\cosh(\mu-2x)}{\cosh\mu}-\frac{\cosh 2(\mu-2x)}{\cosh 2\mu}+\&c.\right\};$$

whence, since the first series on the right hand side$=\frac{1}{4}\pi$, when x is positive and less than μ,

$$\frac{\sin z}{\sinh\frac{1}{2}\nu}+\frac{\sin 3z}{\sinh\frac{3}{2}\nu}+\&c.=\frac{\mu}{2\pi}\left\{1-2\frac{\cosh(\mu-2x)}{\cosh\mu}+2\frac{\cosh 2(\mu-2x)}{\cosh 2\mu}-\&c.\right\}$$

and

$$\tanh x-\tanh(x-\mu)-\tanh(x+\mu)+\&c.=1-2\frac{\cosh(\mu-2x)}{\cosh\mu}+2\frac{\cosh 2(\mu-2x)}{\cosh 2\mu}-\&c.$$

The other transformations to which the method of this section leads are

$$\coth x-\coth(x-\mu)-\coth(x+\mu)+\&c.=1+2\frac{\cosh(\mu-2x)}{\cosh\mu}+2\frac{\cosh 2(\mu-2x)}{\cosh 2\mu}+\&c.,$$

$$\operatorname{cosech} x-\operatorname{cosech}(x-\mu)-\operatorname{cosech}(x+\mu)+\&c.=2\frac{\cosh(\frac{1}{2}\mu-x)}{\cosh\frac{1}{2}\mu}+2\frac{\cosh 3(\frac{1}{2}\mu-x)}{\cosh\frac{3}{2}\mu}+\&c.,$$

$$\operatorname{cosech} x+\operatorname{cosech}(x-\mu)+\operatorname{cosech}(x+\mu)+\&c.=2\frac{\sinh(\frac{1}{2}\mu-x)}{\sinh\frac{1}{2}\mu}+2\frac{\sinh 3(\frac{1}{2}\mu-x)}{\sinh\frac{3}{2}\mu}+\&c.,$$

$$\operatorname{sech} x+\operatorname{sech}(x-\mu)+\operatorname{sech}(x+\mu)+\&c.=2\frac{\cosh(\frac{1}{2}\mu-x)}{\sinh\frac{1}{2}\mu}-2\frac{\cosh 3(\frac{1}{2}\mu-x)}{\sinh\frac{3}{2}\mu}+\&c.,$$

which can be readily verified by ordinary algebra in the manner explained above. In all these identities x must be positive and less than μ.

§ 16. It only remains to apply the methods of §§ 11 and 15 to the identities (33) and (36), which differ from the others by relating to non-periodic functions. Employing the method of § 11, we have

$$\tanh x=\frac{2x}{x^2+(\frac{1}{2}\pi)^2}+\frac{2x}{x^2+(\frac{3}{2}\pi)^2}+\frac{2x}{x^2+(\frac{5}{2}\pi)^2}+\&c.,$$

$$\tanh(x-\mu)=\frac{2(x-\mu)}{(x-\mu)^2+(\frac{1}{2}\pi)^2}+\frac{2(x-\mu)}{(x-\mu)^2+(\frac{3}{2}\pi)^2}+\frac{2(x-\mu)}{(x-\mu)^2+(\frac{5}{2}\pi)^2}+\&c.,$$

$$\tanh(x+\mu)=\frac{2(x+\mu)}{(x+\mu)^2+(\frac{1}{2}\pi)^2}+\frac{2(x+\mu)}{(x+\mu)^2+(\frac{3}{2}\pi)^2}+\frac{2(x+\mu)}{(x+\mu)^2+(\frac{5}{2}\pi)^2}+\&c.,$$

$$\cdots\cdots\cdots\cdots;$$

whence

$$\begin{aligned}\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c.&=\frac{4\pi}{\mu}\left\{e^{-\nu}\sin 2z+e^{-2\nu}\sin 4z+\&c.\right\}\\&+\frac{4\pi}{\mu}\left\{e^{-3\nu}\sin 2z+e^{-6\nu}\sin 4z+\&c.\right\}\\&+\frac{4\pi}{\mu}\left\{e^{-5\nu}\sin 2z+e^{-10\nu}\sin 4z+\&c.\right\}\\&+\cdots\cdots\\&=\frac{4\pi}{\mu}\left(\frac{e^{-\nu}}{1-e^{-2\nu}}\sin 2z+\frac{e^{-2\nu}}{1-e^{-4\nu}}\sin 4z+\&c.\right)\\&=\frac{2\pi}{\mu}\left(\frac{\sin 2z}{\sinh\nu}+\frac{\sin 4z}{\sinh 2\nu}+\&c.\right);\quad\ldots\quad(48)\end{aligned}$$

whereas the true equation is

$$\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c.=\frac{2x}{\mu}+\frac{2\pi}{\mu}\left(\frac{\sin 2z}{\sinh\nu}+\frac{\sin 4z}{\sinh 2\nu}+\&c.\right). \quad . \quad (49)$$

It is well known that if an infinite system of series be summed by rows and by columns, the results need not necessarily be the same; but the above is a striking instance of such a disagreement. We should be prepared for some ambiguity from the observation that although the value of the left-hand side is liable to a change of a unit according as the number of terms retained is even or uneven, yet in the process of transformation no condition whatever with regard to the number of terms in the columns is, or can be, imposed; but we should scarcely expect to obtain an absolutely erroneous result by an apparently definite process.

If the same method be applied to the hyperbolic cotangent, we have

$$\coth x=\frac{1}{x}+\frac{2x}{x^2+\pi^2}+\frac{2x}{x^2+(2\pi)^2}+\&c.,$$

and finally

$$\coth x+\coth(x-\mu)+\coth(x+\mu)+\&c.=\frac{\pi}{\mu}\coth z+\frac{4\pi}{\mu}\left(\frac{\sin 2z}{e^{2\nu}-1}+\frac{\sin 4z}{e^{4\nu}-1}+\&c.\right), \quad . \quad (50)$$

which is also erroneous, the term $\frac{2x}{\mu}$ being omitted on the right-hand side.

The method of § 15, however, yields correct results, for

$$\frac{2\pi}{\mu}\operatorname{cosech}\nu\ \sin 2z=\frac{2}{\mu}\left\{\frac{\pi}{\nu}-\frac{2\frac{\nu}{\pi}}{1^2+\frac{\nu^2}{\pi^2}}+\frac{2\frac{\nu}{\pi}}{2^2+\frac{\nu^2}{\pi^2}}-\&c.\right\}\sin 2z,$$

$$\frac{2\pi}{\mu}\operatorname{cosech}2\nu\sin 4z=\frac{2}{\mu}\left\{\frac{\pi}{2\nu}-\frac{2\,.\,\frac{2\nu}{\pi}}{1^2+\frac{2^2\nu^2}{\pi^2}}+\frac{2\,.\,\frac{2\nu}{\pi}}{2^2+\frac{2^2\nu^2}{\pi^2}}-\&c.\right\}\sin 4z,$$

. . . . ;

whence

$$\frac{2\pi}{\mu}(\operatorname{cosech}\nu\sin 2z+\operatorname{cosech}2\nu\sin 4z+\&c.)$$

$$=\frac{2}{\pi}\left(\sin 2z-\frac{2\sin 2z}{\frac{\mu^2}{\pi^2}+1^2}+\frac{2\sin 2z}{\frac{2^2\mu^2}{\pi^2}+1^2}-\&c.\right)$$

$$+\frac{2}{\pi}\left(\frac{\sin 4z}{2}-\frac{4\sin 4z}{\frac{\mu^2}{\pi^2}+2^2}+\frac{4\sin 4z}{\frac{2^2\mu^2}{\pi^2}+2^2}-\&c.\right)$$

$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$=\frac{2}{\pi}\left\{\frac{\pi}{2}-z-\pi\frac{\sinh\frac{\mu}{\pi}(\pi-2z)}{\sinh\mu}+\pi\frac{\sinh\frac{2\mu}{\pi}(\pi-2z)}{\sinh 2\mu}-\&c.\right\}$$

(by use of the formula $\sin\theta+\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta+\&c.=\frac{1}{2}\pi-\frac{1}{2}\theta$)

$$=1-\frac{2x}{\mu}-2\frac{\sinh(\mu-2x)}{\sinh\mu}+2\frac{\sinh 2(\mu-2x)}{\sinh 2\mu}-\&c.;$$

and therefore

$$\frac{2x}{\mu}+\frac{2\pi}{\mu}\left(\frac{\sin 2z}{\sinh\nu}+\frac{\sin 4z}{\sinh 2\nu}+\&c.\right)=1-2\frac{\sinh(\mu-2x)}{\sinh\mu}+2\frac{\sinh 2(\mu-2x)}{\sinh 2\mu}-\&c.$$

$$=1-2e^{-2x}+2e^{-4x}-\&c.+2\left(\frac{e^{2x}-e^{-2x}}{e^{2\mu}-1}-\frac{e^{4x}-e^{-4x}}{e^{4\mu}-1}+\&c.\right)$$

$$=1-\frac{2}{e^{2x}+1}+2\left\{(e^{2x}-e^{-2x})(e^{-2\mu}+e^{-4\mu}+\&c.)-(e^{4x}-e^{-4x})(e^{-4\mu}+e^{-8\mu}+\&c.)+\&c.\right\}$$

$$=\tanh x+2\left\{\frac{e^{2(x-\mu)}}{1+e^{2(x-\mu)}}-\frac{e^{-2(x+\mu)}}{1+e^{-2(x+\mu)}}+\&c.\right\}$$

$$=\tanh x+\tanh(x-\mu)+\tanh(x+\mu)+\&c.,$$

which is the true formula.

In the same way, since

$$\frac{2\pi}{\mu}\coth\nu\sin 2z=\frac{2}{\mu}\left\{\frac{\pi}{\nu}+\frac{2\frac{\nu}{\pi}}{1^2+\frac{\nu^2}{\pi^2}}+\frac{2\frac{\nu}{\pi}}{2^2+\frac{\nu^2}{\pi^2}}+\&c.\right\}\sin 2z,$$

we find that

$$\frac{2\pi}{\mu}(\coth\nu\sin 2z+\coth 2\nu\sin 4z+\&c.)=-\frac{2x}{\mu}+1+2\frac{\sinh(\mu-2x)}{\sinh\mu}+2\frac{\sinh 2(\mu-2x)}{\sinh 2\mu}+\&c.$$

$$=-\frac{2x}{\mu}+\coth x+\coth(x-\mu)+\coth(x+\mu)+\&c.,$$

which is correct, and agrees with (36).

It is of course easy to assure one's self that (48) cannot be true; for, taking $\mu=\pi$ for simplicity, and differentiating with regard to x or z,

$$\frac{4}{(e^x+e^{-x})^2}+\frac{4}{(e^{x-\pi}+e^{-(x-\pi)})^2}+\frac{4}{(e^{x+\pi}+e^{-(x+\pi)})^2}+\&c.=8\left\{\frac{\cos 2x}{e^{\pi}-e^{-\pi}}+\frac{2\cos 4x}{e^{2\pi}-e^{-2\pi}}+\frac{3\cos 6x}{e^{3\pi}-e^{-3\pi}}+\&c.\right\};$$

and it is evident that if we take $x>\frac{1}{4}\pi$ and $<\frac{3}{4}\pi$ we should have a positive quantity equated to a negative quantity.

I thought it of interest to actually verify numerically the truth of the formulæ (33) and (36) in one or two cases. Working with seven-figure logarithms, and taking $\mu=2$, $x=\frac{1}{2}$, I found that each side of (33) was $=0{\cdot}545188$, and for $\mu=2$, $x=\frac{1}{4}$ that each side was $=0{\cdot}282281$; while for $x=\frac{1}{2}$, $\mu=2$ each side of (36) was $=2{\cdot}07112$, and for $x=\frac{1}{4}$, $\mu=2$ each side was $=4{\cdot}04247$; placing beyond doubt the correctness of (33) and (36).

It is a characteristic property of the identities noticed in this paper that in all cases the series on both sides are convergent whatever may be the values of x and μ. For the actual calculation of the elliptic functions the formulæ (10) to (17) would be preferable to (1) to (8) if the angle of the modulus was very near to 90°, so that q was nearly equal to unity; but as probably the theta functions (or their transformations as in (28)) would always afford the best means of actually calculating the elliptic functions, I have not investigated whether (10) to (17) would present any advantages over the formulæ which result directly from the change of modulus from k to k', as *ex. gr.* the formula at the beginning of § 4, viz.

$$\cos \operatorname{am} u = \frac{\pi}{k\mathrm{K}'}\left\{\frac{1}{e^{z}+e^{-z}} - \frac{r}{1+r}(e^{z}+e^{-z}) + \frac{r^3}{1+r^3}(e^{3z}+e^{-3z}) - \&c.\right\}.$$

§ 17. There are two well-marked classes of identities that are derived from the theory of elliptic functions, viz. pure algebraical identities, in which only one single letter is involved, as *ex. gr.*

$$(1-2q+2q^4-\&c.)^4+(2q^{\frac{1}{4}}+2q^{\frac{9}{4}}+\&c.)^4=(1+2q+2q^4+\&c.)^4,$$

and what may for the sake of distinction be called transcendental identities, viz. in which a function of μ is equated to a function of $\frac{\pi^2}{\mu}$. To this latter class belong the chief identities discussed in this memoir; and if special values be assigned to x such that the left-hand member of the equation is of the same function of μ that the right-hand member is of $\frac{\pi^2}{\mu}$, or, in other words, if the identity is of the form $\phi(\mu)=\phi(\nu)$, where $\mu\nu=\pi^2$, such a result is usually very interesting. The best known identity of this class is

$$\sqrt[4]{\log\frac{1}{q}}\,(\tfrac{1}{2}+q+q^4+q^9+\&c.)=\sqrt[4]{\log\frac{1}{r}}\,(\tfrac{1}{2}+r+r^4+r^9+\&c.)\,; \quad . \quad . \quad (51)$$

but there is another elegant formula of the same kind to which ABEL has drawn attention (Œuvres, t. i. p. 307), viz.

$$\frac{1}{\sqrt[24]{q}}(1+q)(1+q^3)(1+q^5)\ldots=\frac{1}{\sqrt[24]{r}}(1+r)(1+r^3)(1+r^5)\ldots, \quad . \quad . \quad . \quad (52)$$

the relation between q and r being of course

$$\log q\,.\,\log r=\pi^2.$$

It seems probable that all the transcendental formulæ of this latter class can be deduced from the trigonometrical identities in § 11 and at the beginning of § 12 by elementary methods, without the introduction of elliptic-function formulæ; and it is of some interest to verify (52) in this way.

Starting from the formula (23), which may be written

$$-\tfrac{1}{4}\operatorname{cosec} x+\frac{\sin x}{1+e^{\mu}}+\frac{\sin 3x}{1+e^{3\mu}}+\&c.=-\frac{\pi}{2\mu}\left\{\frac{1}{e^{z}-e^{-z}}-\frac{1}{e^{z-\nu}-e^{-(z-\nu)}}-\frac{1}{e^{z+\nu}-e^{-(z+\nu)}}+\&c.\right\},$$

we have, on differentiation with regard to x,

$$\frac{1}{4}\frac{\cos x}{\sin^2 x}+\frac{\cos x}{1+e^{\mu}}+\frac{3\cos 3x}{1+e^{3\mu}}+\&c.=\frac{\pi^2}{2\mu^2}\left\{\frac{e^{z}+e^{-z}}{(e^{z}-e^{-z})^2}-\frac{e^{z-\nu}+e^{-(z-\nu)}}{(e^{z-\nu}-e^{-(z-\nu)})^2}-\frac{e^{z+\nu}+e^{-(z+\nu)}}{(e^{z+\nu}-e^{-(z+\nu)})^2}+\&c.\right\}.$$

Put $x=0$, and

$$\tfrac{1}{4}\frac{\cos x}{\sin^2 x}=\tfrac{1}{4}\cdot\frac{1-\frac{1}{2}x^2}{x^2(1-\frac{1}{3}x^2)}=\frac{1}{4x^2}(1-\tfrac{1}{2}x^2+\tfrac{1}{3}x^2)$$

$$=\frac{1}{4x^2}-\tfrac{1}{24},$$

while

$$\frac{\pi^2}{2\mu^2}\frac{e^z+e^{-z}}{(e^z-e^{-z})^2}=\frac{1}{4x^2}\left(1+\tfrac{1}{2}\frac{\pi^2x^2}{\mu^2}-\tfrac{1}{3}\frac{\pi^2x^2}{\mu^2}\right)$$

$$=\frac{1}{4x^2}+\tfrac{1}{24}\frac{\pi^2}{\mu^2};$$

so that

$$-\tfrac{1}{24}+\frac{1}{1+e^{\mu}}+\frac{3}{1+e^{3\mu}}+\&c.=\tfrac{1}{24}\frac{\pi^2}{\mu^2}-\frac{\pi^2}{\mu^2}\left\{\frac{e^{\nu}+e^{-\nu}}{(e^{\nu}-e^{-\nu})^2}-\frac{e^{2\nu}+e^{-2\nu}}{(e^{2\nu}-e^{-2\nu})^2}+\&c.\right\}$$

$$=\tfrac{1}{24}\frac{\pi^2}{\mu^2}-\frac{\pi^2}{\mu^2}\{(e^{-\nu}+e^{-3\nu})(1+2e^{-2\nu}+3e^{-4\nu}+4e^{-6\nu}+\&c.)$$

$$-(e^{-2\nu}+e^{-6\nu})(1+2e^{-4\nu}+3e^{-8\nu}+4e^{-12\nu}+\&c.)$$

$$+\&c.\qquad\}$$

$$=\tfrac{1}{24}\frac{\pi^2}{\mu^2}-\frac{\pi^2}{\mu^2}\left\{\frac{e^{-\nu}}{1+e^{-\nu}}+\frac{3e^{-3\nu}}{1+e^{-3\nu}}+\frac{5e^{-5\nu}}{1+e^{-5\nu}}+\&c.\right\}$$

$$=\tfrac{1}{24}\frac{\pi^2}{\mu^2}-\frac{\pi^2}{\mu^2}\left\{\frac{1}{1+e^{\frac{\pi^2}{\mu}}}+\frac{3}{1+e^{\frac{3\pi^2}{\mu}}}+\&c.\right\};$$

whence, on integration with regard to μ,

$$-\frac{\mu}{24}-\log(1+e^{-\mu})-\log(1+e^{-3\mu})-\&c.=-\tfrac{1}{24}\frac{\pi^2}{\mu}-\log(1+e^{-\frac{\pi^2}{\mu}})-\log(1+e^{-\frac{3\pi^2}{\mu}})-\&c.$$

$$+\text{const.},$$

viz.

$$e^{\frac{\mu}{24}}(1+e^{-\mu})(1+e^{-3\mu})\ldots=\text{C}\,.\,e^{\frac{\pi^2}{24\mu}}(1+e^{-\frac{\pi^2}{\mu}})(1+e^{-\frac{3\pi^2}{\mu}})\ldots,$$

and $\text{C}=1$, as is seen by putting $\mu=\pi$; so that (52) is established.

The other identity (51), or rather the generalization of it,

$$e^{-x^2}+e^{-(x-\mu)^2}+e^{-(x+\mu)^2}+\&c.=\frac{\sqrt{\pi}}{\mu}\left\{1+2e^{-\frac{\pi^2}{\mu^2}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{4\pi^2}{\mu^2}}\cos\frac{4\pi x}{\mu}+\&c.\right\}\qquad(53)$$

(which is much more difficult to prove by elementary methods than any of the identities discussed in this paper), I deduced by algebraical processes from the equation in § 12, viz. from

$$\frac{a}{x^2+a^2}+\frac{a}{(x-\mu)^2+a^2}+\frac{a}{(x+\mu)^2+a^2}+\&c.=\frac{\pi}{\mu}\left\{1+2e^{-\frac{2\pi a}{\mu}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{4\pi a}{\mu}}\cos\frac{4\pi x}{\mu}+\&c.\right\},\qquad(54)$$

in the Philosophical Magazine for June 1874 (ser. 4, vol. xlvii. p. 437 *et seq.*); but it perhaps is worth while to note here what is the most natural way of obtaining it from

(54), viz. by help of the theorems

$$\left[e^{-n^2\frac{d^2}{da^2}}\frac{a}{x^2+a^2}\right]_{a=0}=\frac{\sqrt{\pi}}{2n}e^{-\frac{x^2}{4n^2}}, \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$\left[e^{-n^2\frac{d^2}{da^2}}e^{-ax}\right]_{a=0}=e^{-n^2x^2}; \quad . \quad . \quad . \quad . \quad . \quad (56)$$

whence, operating on (54) with $e^{-n^2\frac{d^2}{da^2}}$, and making $a=0$, we have at once

$$\frac{\sqrt{\pi}}{2n}\left\{e^{-\frac{x^2}{4n^2}}+e^{-\frac{(x-\mu)^2}{4n^2}}+e^{-\frac{(x+\mu)^2}{4n^2}}+\&c.\right\}=\frac{\pi}{\mu}\left\{1+2e^{-\frac{4n^2\pi^2}{\mu^2}}\cos\frac{2\pi x}{\mu}+2e^{-\frac{16n^2\pi^2}{\mu^2}}\cos\frac{4\pi x}{\mu}+\&c.\right\},$$

which is (53) if we take $n=\frac{1}{2}$.

Of the two lemmas (55) and (56) the truth of the second is seen at once, for

$$e^{-n^2\frac{d^2}{da^2}}e^{-ax}=\left(1-n^2\frac{d^2}{da^2}+\frac{1}{1.2}n^4\frac{d^4}{da^4}-\&c.\right)e^{-ax}$$

$$=\left(1-n^2x^2+\frac{1}{1.2}n^4x^2-\&c.\right)e^{-ax}$$

$$=e^{-n^2x^2-ax};$$

and (55) is easily established, since a being put $=0$ after the performance of the differentiations,

$$e^{-n^2\frac{d^2}{da^2}}\frac{a}{x^2+a^2}=e^{-n^2\frac{d^2}{da^2}}\int_0^\infty e^{-au}\cos xu\,du=\int_0^\infty e^{-n^2u^2}\cos xu\,du$$

$$=\frac{\sqrt{\pi}}{2n}\cdot e^{-\frac{x^2}{4n^2}}.$$

But the investigation is not elementary; and if we assume a knowledge of the integral

$$\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx=\frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{a^2}},$$

we may as well apply it directly to prove (53) by FOURIER's theorem as explained in § 7, or employ it as SCHELLBACH has done ('Die Lehre von den elliptischen Integralen &c.,' 1864, p. 30). It does not seem to be easy to establish (55) without the aid of an integral; for, expanding in ascending powers of x, we have to show that when $a=0$,

$$e^{-n^2\frac{d^2}{da^2}}\left(\frac{1}{a}-\frac{x^2}{a^3}+\frac{x^4}{a^5}-\&c.\right)=\frac{\sqrt{\pi}}{2n}\left(1-\frac{x^2}{4n^2}+\frac{x^4}{32n^4}-\&c.\right);$$

and, taking the first term only, although we see at once that

$$e^{-n^2\frac{d^2}{da^2}}\cdot\frac{1}{a}=e^{-n^2\frac{d^2}{da^2}}\int_0^\infty e^{-au}du=\int_0^\infty e^{-n^2u^2}du=\frac{\sqrt{\pi}}{2n},$$

yet

$$e^{-n^2\frac{d^2}{da^2}}\cdot\frac{1}{a}=\frac{1}{a}-\frac{1.2.n^2}{a^3}+\frac{1.2.3.4n^4}{a^5}-\&c.,$$

which is divergent, and cannot apparently by any simple method be so transformed that its value when $a=0$ may be evident, without the intervention of an integral. Thus the method depending upon (55), though more direct, is not so elementary as that described in the Philosophical Magazine.

It is curious that all the formulæ of the form

$$\phi x \pm \phi(x-\mu) \pm \phi(x+\mu) + \&c. = \text{series of sines or cosines}$$

which can be obtained by definite integrals, and which possess any interest, should be in reality elliptic-function identities. Of course every result that can be derived from these identities by differentiation, by multiplication by a factor and integration, &c., can as a rule be obtained directly from an integral, which integral itself would arise from a similar treatment of the original integral. This is true of the identities in the Philosophical Magazine, ser. 4, vol. xlii. pp. 422 *et seq.* (December 1871); and, for example, such an integral as

$$\int_0^\infty \frac{e^{-x^2}\cos 2bx}{a^2+x^2}\,dx = \frac{\sqrt{\pi}}{2a}\,e^{a^2}\{e^{-2ab}\operatorname{erfc}(a-b)+e^{2ab}\operatorname{erfc}(a+b)\} \quad . \quad . \quad . \quad (57)$$

(where $\operatorname{erfc} x = \int_x^\infty e^{-x^2}dx$) would give rise to identities which, however, could be deduced from (28) and (53) by a similar process to that by which (57) can be derived from

$$\int_0^\infty e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}\,e^{-b^2}.$$